From Syllogism to Common Sense

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Quantification

First-Order Logic
Description Logics
Modality and Quantification

Lecture 10

Outline of Lecture 10

- Basics of First-order logic
- A sketch of Description Logics and the relation to First-order logic and Modal Logic
- Combining quantifiers with modalities
- Counterpart Theory

First-Order Logic (FOL)

- FOL is an expressive, general purpose language
- With historical roots in
  - Aristotelian Syllogisms
    - e.g. conclusions inferred from two (quantificational) premises
  - Boole's logic
    - e.g. the basic algebraic rules governing conjunction, negation, etc. (1854)
  - Frege's Begriffsschrift
    - a fully formal notation for logic encompassing modern first-order (1879)
  - Peirce's logical investigations
    - e.g. the distinction between first- and second-order quantifier (1885)
- Important historically in axiomatising foundational theories in mathematics

First-Order Logic

- The `**axiomatic method**':
  - completely abstract description of a domain by specification of what objects exist and what their properties are, e.g.:
    - **Abstract Algebra** (groups, fields, vector spaces) (e.g. Galois 1832)
    - **Euclid's Geometry** as axiomatised by David Hilbert (1899)
    - **Set Theory, e.g.** ZFC (1908-1930) or NBG (1920s-1940s)
  - also applied in
    - specification of software
    - logic programming
    - axiomatisation of upper/foundational ontologies

FOL: More History

- Gödel's **completeness theorem**, proved by Kurt Gödel in 1929
- **Undecidability**: Alonzo Church and Alan Turing in 1936 and 1937, respectively, gave a negative answer to the Entscheidungsproblem posed by David Hilbert in 1928.
- **Expressivity:** By the Löwenheim–Skolem theorem, there is no FOL theory that has as its unique model the natural numbers.
- **Many-sorted** first-order logic is mostly like standard first-order logic, but with a set **S** of **sorts** and correspondingly sorted constant, function, and predicate symbols.

Many-Sorted First-Order Logic: Syntax

- **Non-logical symbols (signatures)**
  - a signature Σ is a triple <S, F, P> with
    - S a set of **sorts,** and S* the set of words over S
    - for each $w \in S^*$, a subset $F_{w,s} \subseteq F$ of **function symbols**
    - for each $w \in S^*$, a subset $P_w \subseteq P$ of **predicate symbols**
    - constants of sort s are the nullary functions in $F_{\varepsilon,s}$
- **Logical symbols**
  - a set X_s of variables for each sort s.
  - the Boolean operators, conjunction, negation, etc.
  - the identity symbol = and the quantifiers 'for all' ∀ and 'exists' ∃.
- **Formulae** are constructed in the usual way respecting typing

Many-Sorted First-Order Logic: Semantics

- Given a signature Σ = <S, F, P>, a Σ-model $\mathcal{M}$ consists of:
  - A carrier set $M_s \neq \emptyset$ for each sort $s \in S$
  - A function $f^{\mathcal{M}}_{w,s} : M_{s_1} \times \ldots \times M_{s_n} \to M_s$ for each $f \in F_{w,s}$, where $w = s_1 \cdots s_n$.
    In particular, for a constant, this is just an element of M_s.
  - A relation $p^{\mathcal{M}}_w \subseteq M_{s_1} \times \ldots \times M_{s_n}$ for each $p \in P_w, w = s_1 \cdots s_n$.

'Standard' FOL has simply just one sort (single-sorted).

Natural Language Examples

- "Every person who lives in Bremen lives in Germany"
  - Pick a unary predicate **Person**
  - Two constants **Bremen** and **Germany**
  - A binary relation **lives-in**
- Formalise as:
  - $\forall$ x . Person (x) $\wedge$ lives-in(x, Bremen) $\rightarrow$ lives-in(x, Germany)
- Alternative formalisations:
  - Axiomatise: Cities, Countries, Containment (Parthood)
- **Limitations:** Quantification over predicates, modalities, constructions such as 'terribly small', 'walking quickly', etc.

FOL Example: Mereology in Dolce

- **Parthood** is a **partial order,** i.e. a reflexive, antisymmetric, transitive **binary relation**
- **Generic mereology** for sort s, and mereology for sorts T, S, PD

spec GenMereology [**sort** s] =
GenParthood [**sort** s]
then
preds $PP(x, y\colon s) \Leftrightarrow P(x, y) \wedge \neg P(y, x);$
$O(x, y\colon s) \Leftrightarrow \exists z\colon s \bullet P(z, x) \wedge P(z, y);$
$At(x\colon s) \Leftrightarrow \neg \exists y\colon s \bullet PP(y, x);$
then
$\forall x, y\colon s$
$\bullet \neg P(x, y) \Rightarrow (\exists z\colon s \bullet P(z, x) \wedge \neg O(z, y))$
$\bullet \exists z\colon s \bullet At(z) \wedge P(z, x)$
then %implies
$\forall x, y, z, z'\colon s$
$\bullet (\forall z'\colon s \bullet At(z') \Rightarrow P(z', x) \Rightarrow P(z', y)) \Rightarrow P(x, y)$
$\bullet (\forall z\colon s \bullet O(z, x) \Leftrightarrow O(z, y)) \Rightarrow x = y$
end

spec Mereology =
Primitives
then
GenMereology [**sort** T]
then
GenMereology [**sort** S]
then
GenMereology [**sort** PD]
end

Description Logics

- DLs focus on the representation of terminological knowledge:
  - formalise the basic **terminology** adopted in an application
- Terminologies are formalised as a collection of **concepts** and **relations**
  - e.g. '*Course*', '*Lecturer*', and '*gives_course*', '*attends_lecture*'
- DL knowledge bases define basic concepts and give relationships between them in the form of subsumptions

The Description Logic $\mathcal{ALC}$

- Atomic symbols:
  - concept names (unary predicates): $A, B, C, D, \ldots$
  - role names (binary predicates): $R, S, T, \ldots$
- Concept constructors:

		Manchester Syntax (Hets)
Top/Bottom	$\top$, $\bot$	Thing , Nothing
negation	$\neg C$	not C
conjunction	$C \sqcap D$	C and D
disjunction	$C \sqcup D$	C or D
existential restriction	$\exists R.C$	R some C
value restriction	$\forall R.C$	R only C

- Complex concepts: $\neg(A \sqcup \exists R.(\forall S.B \sqcap \neg C))$
- For example: Human $\sqcap$ $\exists$ Lives-in Bremen $\sqsubseteq$ $\exists$ Lives-in Germany

Semantics of $\mathcal{ALC}$

- A model is a pair <W, I> where W is a set and I an **interpretation function**
  - assigning subsets of W to concept names $A, B, C, D, \ldots$
  - subsets of W × W to role names $R, S, T, \ldots$
- Concept constructors:
  - W, empty set ∅ — $\top\ ,\ \bot$
  - set complement — $\neg C$
  - set intersection — $C \sqcap D$
  - set union — $C \sqcup D$
  - { v ∈ W ∣ ∃ w ∈ W . v R w ∧ C(w) } — $\exists R.C$
  - { v ∈ W ∣ ∀ w ∈ W . v R w → C(w) } — $\forall R.C$

Example: Pizzas

- Here is a small excerpt from the pizza ontology:

```
VegetarianPizza      ⊑ Pizza
MagheritaPizza       ⊑ Pizza
TomatoTopping        ⊑ VegetableTopping
MozzarellaTopping    ⊑ CheeseTopping
VegetarianPizza      ≡ ∀ hasTopping (VegetableTopping ⊔ CheeseTopping)
MagheritaPizza       ⊑ ∃ hasTopping MozarellaTopping ⊓
                        ∃ hasTopping TomatoTopping ⊓
                        ∀ hasTopping (MozzarellaTopping ⊔ TomatoTopping)
```

- It follows that the following is true in all models of this Tbox:

`MagheritaPizza ⊑ VegetarianPizza`

OWL: DL $\mathcal{SROIQ}$

- The web ontology language OWL uses the logic $\mathcal{SROIQ}$, adding e.g.:

	(Hets) Manchester syntax	Semantics
$\phi ::= \ldots$		
$\mid R_1;\ldots;R_n \sqsubseteq R$	`ObjectProperty: R SuperPropertyOf R_1; ... ;R_n`	$R_1^{\mathcal{I}} \circ \ldots \circ R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\mid Dis(R_1, R_2)$	`ObjectProperty: R_1 Disjoint R_2`	$R_1^{\mathcal{I}} \cap R_2^{\mathcal{I}} = \emptyset$
$\mid Ref(R)$	`ObjectProperty: R Reflexive`	$\forall x \in \Delta^{\mathcal{I}}.R^{\mathcal{I}}(x,x)$
$\mid Irr(R)$	`ObjectProperty: R Irreflexive`	$\forall x \in \Delta^{\mathcal{I}}.\neg R^{\mathcal{I}}(x,x)$
$\mid Asy(R)$	`ObjectProperty: R Asymetric`	$\forall x, y \in \Delta^{\mathcal{I}}\ R(x,y) \rightarrow R(y,x)$

where $R \circ S = \{(x,z) \mid \exists y.(x,y) \in R, (y,z) \in S\}$

The new concept $\exists R.Self$ *with* $(\exists R.Self)^{\mathcal{I}} = \{x \mid x \in \Delta^{\mathcal{I}}, (x,x) \in R^{\mathcal{I}}\}$ *and the universal role* U *with* $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

- For more on DLs and OWL, see e.g.:
  - http://dl.kr.org/courses.html
  - http://semantic-web-book.org/page/ESSLLI_2009

Logic Translations

- How do we move from one logic to another?
  - change of syntax
  - change of semantics
- Requirements
  - preserve the meaning of the original formalisation
  - models of the original formulas should be 'obtainable' from the models of the translated formulas

Example: The Standard Translation

- The standard translation T from $\mathcal{ALC}$ to FOL maps
  - Concepts names A ➜ unary predicates P_A
  - Role names R ➜ binary predicates P_R
  - Object names a ➜ constants c_a
- and uses the following translation rules for complex concepts:

$$\begin{aligned} T^x(A) &= P_A(x) \\ T^x(\neg C) &= \neg T^x(C) \\ T^x(C \sqcap D) &= T^x(C) \wedge T^x(D) \\ T^x(C \sqcup D) &= T^x(C) \vee T^x(D) \\ T^x(\exists R.C) &= \exists y.P_R(x,y) \wedge T^y(C) \\ T^x(\forall R.C) &= \forall y.P_R(x,y) \rightarrow T^y(C) \end{aligned}$$

- Concepts correspond to formulas with **exactly one free variable**.
- T^y is just like T^x with x and y interchanged
- Note that **two variables** are enough for the translation

Modal Logic and DL

- Modal (propositional) logic adds sentential operators like 'possibly', or 'necessarily', to propositional logic:

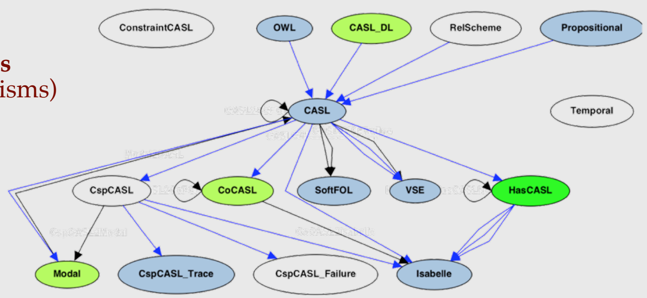
$\mathcal{ALC} \cong$ multimodal logic K

- The correspondence on the right gives rise to a logic translation.

$\mathcal{ALC}$	K
concept C	formula φ
atomic concept A	propositional variable p
$C \sqcap D$	$\varphi \wedge \psi$
$C \sqcup D$	$\varphi \vee \psi$
$\neg C$	$\neg\varphi$
$\exists R.C$	$\Diamond_R \varphi$
$\forall R.C$	$\Box_R \varphi$
$(\Delta, (C^\mathcal{I})_{C\in\mathbf{C}}, (R^\mathcal{I})_{R\in\mathbf{R}})$	Kripke model with set of worlds (Δ), accessibility relations $(R^\mathcal{I})$ and interpretation of propositional variables $C^\mathcal{I}$

Heterogeneous Ontologies

- In order to systematically link ontological modules formulated in different formalisms we need to:
  - fix a **logic graph**
  - give **logic translations** (institution co-morphisms)



Modality and Quantification

Basic Problems
Barcan Formulae
Lambda Abstraction
Counterparts

Outline

- Modality, Quantification, and Identity (Aristotle, Modal Syllogistic)
- De Re/De Dicto, and Lambda Abstraction (Stalnaker, R. Thomason 1968, Fitting 1998)
- Kripke Semantics for QML (Kripke 1963)
- FOL Counterpart Theory as Semantics for QML (David Lewis 1968)

Combining Modality and Quantifiers

Example: combining modality, quantification, and identity

- All humans are necessarily mortal.
- The number of planets is necessarily 9.
- Necessarily, the number of planets is 9.
- The president of the US someday won't be the president of the US.
- The president of the US might not have been Barack Obama.
- I could have been quite unlike what I in fact am.

Semantic complexities: necessary properties, de de, de dicto, identity across worlds, persistence over time, counterfactuality, dissimilar counterparts, etc.

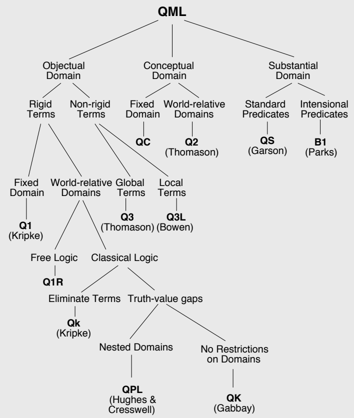
Modality, Quantification, and Identity

Items that can be varied according to universal logic:
signatures - grammar - models - satisfaction

- **Signatures:** (non-logical symbols) propositions; predicates; functions, constants, terms.
- **Grammar:** (logical symbols) variables and quantifiers; modalities; identity symbol; lambda abstraction; well-formed expressions (formulae); substitution.
- **Models:** possible world; domains of discourse; accessibility (counterpart relations) ; object (individual)
- **Satisfaction:** vary the truth conditions for quantifiers; vary conditions for identity statements, etc.

Modality, Quantification, and Identity:

Garson's Forest (1984)



A combination of 2 (or 3) logical theories

- modal predicate logic
- quantified modal logic
- first-order modal logic
- first-order intensional logic, etc.

Possible Objects and Quantification

Objects

- Worldbound or not
- Constant or Varying
- objects/individual concepts/traces of objects/counterparts

Quantification

- Existence property: **E**(x)
- Vary the base logic: classical or free quantification, etc.
- Quantify over what entities?

E.g.: in **free logic**:

$$P(a) \not\vDash \exists x.P(x)$$

but

$$P(a), E(a) \vDash \exists x.P(x)$$

Define **actualist** quantifiers from **possibilist** ones:

$$\forall x.\varphi(x) := \bigwedge x.(E(x) \rightarrow \varphi(x))$$

$$\exists x.\varphi(x) := \bigvee x.(E(x) \wedge \varphi(x))$$

(Simplest) Kripke Semantics for MPL

- Standard frames: a set W of possible worlds, an accessibility R between worlds
- A domain D of (possible individuals), the same for each w in W
- Valuation val(x) assigns (rigidly) values in D to variables x
- Standard quantification theory over D, but interpretations of predicates can vary between worlds
- Formulae are constructed in the obvious way combining FOL grammar with modalities as new sentence operators

Models are: $M = \langle W, R, D, val \rangle$

Satisfaction is

$$\langle M, v \rangle \models \forall x.\phi(x) \text{ iff for all } d \in D \text{ we have } \langle M(d/x), v \rangle \models \phi(x)$$

$$\langle M, v \rangle \models \Box\phi(x) \text{ iff for all } w \in W \text{ such that } vRw : \langle M, w \rangle \models \phi(x)$$

De Re and De Dicto

- Basic distinction in modal predicate logic since medieval times:
  - **de re:** res = object:
    a property is true of an object necessarily:
    - example: the number 9 is necessarily odd
  - **de dicto:** dicto = statement:
    a proposition is necessarily true:
    - example: necessarily, 2+2 = 4

Semantics vs Syntax

- Modelling insufficiencies can be addressed by both semantic and syntactic modifications:
- Syntax examples: lambda abstraction to disambiguate scope of terms (next slide);
- Semantic examples: changing truth definition, e.g. interpret objects as wordlines (individual concepts) with traces etc.

Lambda Abstraction

Skolemisation of $\Box(\exists x)\phi(x)$

necessitates non-rigid constants in $\Box\phi(c)$

Skolemisation of both $\Box(\exists x)\phi(x)$, $(\exists x)\Box\phi(x)$ yields $\Box\phi(c)$

Distinguish: $\Box(\lambda x.\phi(x))(c)$

$(\lambda x.\Box\phi(x))(c)$

Lambda Abstraction

Example: Quine 53, Three Grades of Modal Involvement

Necessarily true: '9 greater or equal 7'

Contingently true: 'number of planets' is greater or equal 7

- $\langle\lambda x.\Box(x \geq 7)\rangle(t)$
- $\Box\langle\lambda x.x \geq 7\rangle(t)$

Alternatively: make 9 **rigid**, but 'number of planets' **non-rigid** $\longrightarrow$ Assymetry between variables and constants

The Barcan Formulae

$$\forall x.\Box P(x) \to \Box\forall x.P(x)$$

- If everything is necessarily P, then it is necessary that everything is P.

By **duality** this is equivalent to

$$\Diamond\exists x.P(x) \to \exists x.\Diamond P(x)$$

- Validity of BF implies that all objects which exist in every possible world (accessible to the actual world) exist in the actual world, i.e. that domains cannot grow. This thesis is sometimes known as **actualism**, i.e. that there are no merely possible individuals.

The Converse Barcan Formulae

$$\Box\forall x.P(x) \to \forall x.\Box P(x)$$

- If it is necessary that everything is P, then everything is necessarily P.

By **duality** this is equivalent to

$$\exists x.\Diamond P(x) \to \Diamond\exists x.P(x)$$

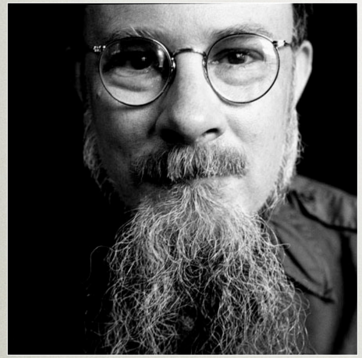
BF: nothing comes into existence

CBF: nothing goes out of existence

Existence, Symmetry and Barcan

- In the constant domain Kripke semantics, both BF and CBF are valid schemas.
- But we can introduce an 'existence predicate' and simulate varying domains (and falsify BF/CBF).
- In symmetric frames, BF and CBF are equivalent.
  - Need to introduce 'varying domain semantics':
  - Tutorial next week

Standard CT as Semantics for MPL



Primitive Predicates

- $W(x)$ *(x ist eine mögliche Welt.)*
- $I(x,y)$ *(x liegt in der möglichen Welt y.)*
- $A(x)$ *(A ist die* aktuale *mögliche Welt.)*
- $C(x,y)$ *(x ist ein Counterpart von y.)*

David Lewis 1968:
Counterpart Theory and Quantified Modal Logic

Standard CT as Semantics for MPL

Postulates

P1 $\forall x \forall y (I(x,y) \rightarrow W(y))$
(„Nothing is in anything except a world")

P2 $\forall x \forall y \forall z (I(x,y) \wedge I(x,z) \rightarrow y \doteq z)$
(„Nothing is in two worlds")

P3 $\forall x \forall y (C(x,y) \rightarrow \exists z I(x,z))$
(„Whatever is a counterpart is in a world")

P4 $\forall x \forall y (C(x,y) \rightarrow \exists z I(y,z))$
(„Whatever has a counterpart is in a world")

P5 $\forall x \forall y \forall z (I(x,y) \wedge I(z,y) \wedge C(x,z) \rightarrow x \doteq z)$
(„Nothing is a counterpart of anything else in its world")

P6 $\forall x \forall y (I(x,y) \rightarrow C(x,x))$
(„Anything in a world is a counterpart of itself")

P7 $\exists x (W(x) \wedge \forall y (I(y,x) \leftrightarrow A(y)))$
(„Some world contains all and only actual things")

P8 $\exists x A(x)$
(„Something is actual")

The actual world $@ := \iota x. \forall y (I(y,x) \leftrightarrow A(y)).$

Standard CT as Semantics for MPL

The standard translation of: $\Diamond \phi(\alpha_1, \ldots, \alpha_n)$ true at beta

$\Diamond \phi(\alpha_1, \ldots, \alpha_n))^\beta$

$$\exists \beta_1 \exists \gamma_1 \ldots \exists \gamma_n (W(\beta_1) \wedge I(\gamma_1, \beta_1) \wedge C(\gamma_1, \alpha_1) \wedge \ldots$$
$$\ldots \wedge I(\gamma_n, \beta_1) \wedge C(\gamma_n, \alpha_n) \wedge \phi^{\beta_1}(\gamma_1, \ldots, \gamma_n))$$

Standard CT as Semantics for MPL

Non valid principles:

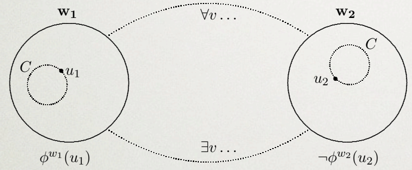
1. $\Box\phi \to \Box\Box\phi$ (“Becker’s Prinzip”)
2. $\phi \to \Box\Diamond\phi$ (“Brouwer’s Prinzip”)
3. $(x \doteq y) \to \Box(x \doteq y)$ (x verschieden von y)
4. $(x \not\doteq y) \to \Box(x \not\doteq y)$ (x verschieden von y)
5. $\forall x\Box\phi(x) \to \Box\forall x\phi(x)$ (Barcan Formeln)
6. $\exists x\Box\phi(x) \to \Box\exists x\phi(x)$
7. $\Box\exists x\phi(x) \to \exists x\Box\phi(x)$

Valid:

$\Box\forall x\phi(x) \;\to\; \forall x\Box\phi(x)$

Failure of Box Distribution in CT

Simultaneous quantification over worlds and individuals



(i) $\mathcal{M} \models (\Box(\phi(x) \wedge \forall x\phi(x)))^{w_1}$

(ii) $\mathcal{M} \models (\Diamond\exists x\neg\phi(x))^{w_1}$, d.h.

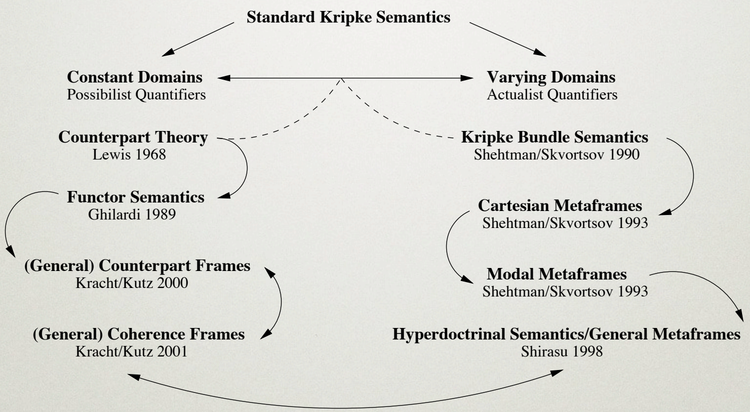
(iii) $\mathcal{M} \models (\neg\Box\forall x\phi(x))^{w_1}$.

Stand. Trans. for $(\Box\chi(x))^w$ is $\forall v\forall y(W(v) \wedge I(y,v) \wedge C(y,x) \to \chi^v(y))$

Tautology: $(\phi(x) \wedge \forall x\phi(x)) \to \forall x\phi(x)$

Box distribution applied: $\Box(\phi(x) \wedge \forall x\phi(x)) \to \Box\forall x\phi(x)$

Generalised Semantics



General Completeness: Balanced Standard Kripke Semantics

- Extend the standard Kripkean semantics for MPL and constant domains as follows:
- Add an equivalence relation E such that:
  - v E w implies P(x1,...xn) is interpreted in the same way (**world-mirrors**).
  - if v R v1 and v E w, then there is a w1 such that w R w1 and v1 E w1 (**mirrored worlds upwards indistinguishable**).
  - interpret identity as equivalence: equivalent objects must be indistinguishabe (**equivalentiality**).

Competing Ontologies

	poly-counterpart	balanced standard
accessibility	relations between individuals	relations between worlds
objects	worldbound individuals	global universe of objects
identity	between individuals	equivalence between objects
Predication	locally	globally (but admissible)
Haecceitism	No	Yes, world-mirrors

Cost of Generality:

- Balanced standard frames assume
  - a notion of haecceitism, identifying twin worlds
  - replace numerical identity by 'admissible' equivalence
  - equivalential predication
- Compare many-worlds interpretation of Quantum Mechanics

Talk Announcement

- Rotunde, Fr. 15:30
- Prof. Jean-Yves Beziau
- "The Power of the Hexagon"

Literature

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