The Complexity of Satisfiability for Sub-Boolean Fragments of $\mathcal{ALC}$

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Abstract. The standard reasoning problem, concept satisfiability, in the basic description logic $\mathcal{ALC}$ is PSPACE-complete, and it is EXPTIME-complete in the presence of unrestricted axioms. Several fragments of $\mathcal{ALC}$, notably logics in the $\mathcal{FL}$, $\mathcal{EL}$, and DL-Lite families, have an easier satisfiability problem; sometimes it is even tractable. All these fragments restrict the use of Boolean operators in one way or another. We look at systematic and more general restrictions of the Boolean operators and establish the complexity of the concept satisfiability problem in the presence of axioms. We separate tractable from intractable cases.

1 Introduction

Standard reasoning problems of description logics, such as satisfiability or subsumption, have been studied extensively. Depending on the expressivity of the logic and the reasoning problem, the complexity of reasoning for DLs ranging from logics below the basic description logic $\mathcal{ALC}$ to the OWL DL standard $\mathcal{SROIQ}$ is between tractable and NEXPTIME.

For $\mathcal{ALC}$, concept satisfiability is PSPACE-complete [26] and, in the presence of unrestricted axioms, it is EXPTIME-complete due to the correspondence with propositional dynamic logic [24, 28, 14]. Since the standard reasoning tasks are interreducible in the presence of all Boolean operators, subsumption has the same complexity.

Several fragments of $\mathcal{ALC}$, such as logics in the $\mathcal{FL}$, $\mathcal{EL}$ or DL-Lite families, are well-understood. They often restrict the use of Boolean operators, and it is known that their reasoning problems are often easier than for $\mathcal{ALC}$. For instance, concept subsumption with respect to acyclic and cyclic TBoxes, and even with GCIs is tractable in the logic $\mathcal{EL}$, which allows only conjunctions and existential restrictions, [4, 9], and it remains tractable under a variety of extensions such as nominals, concrete domains, role chain inclusions, and domain and range restrictions [5, 6]. However, the presence of universal quantifiers breaks tractability: Subsumption in $\mathcal{FL}_0$, which allows only conjunction and universal restrictions, is coNP-complete [21] and increases to PSPACE-complete with respect to cyclic TBoxes [3, 18] and to EXPTIME-complete with GCIs [5, 17]. In [12, 13], concept satisfiability and subsumption for several logics below and above $\mathcal{ALC}$ that extend $\mathcal{FL}_0$ with disjunction, negation and existential restrictions and other features, is shown to be tractable, NP-complete, coNP-complete or PSPACE-complete. Subsumption in the presence of general axioms is EXPTIME-complete in logics containing both existential and universal restrictions plus conjunction or disjunction [15], as well as in $\mathcal{AL}$, where only conjunction, universal restrictions and unqualified existential restrictions are allowed [11]. In DL-Lite, where atomic negation, unqualified existential and universal restrictions, conjunctions and inverse roles are allowed, satisfiability of ontologies is tractable [10]. Several extensions of DL-Lite are shown to have tractable and NP-complete satisfiability problems in [1, 2].

This paper revisits restrictions to the Boolean operators in $\mathcal{ALC}$. Instead of looking at one particular subset of $\{\wedge, \vee, \neg\}$, we are considering all possible sets of Boolean operators, including less commonly used operators such as the binary exclusive or $\oplus$. Our aim is to find for *every* possible combination of Boolean operators whether it makes satisfiability of the corresponding restriction of $\mathcal{ALC}$ hard or easy. Since each Boolean operator corresponds to a Boolean function—*i.e.*, an n-ary function whose arguments and values are in $\{\bot, \top\}$—there are infinitely many sets of Boolean operators determining fragments of $\mathcal{ALC}$. The complexity of the corresponding concept satisfiability problems without theories, which are equivalent to the satisfiability problems for the corresponding fragments of multimodal logic, has already been classified in [16]: it is PSPACE-complete if at least the ternary operator $x \wedge (y \vee z)$ and the constant $\bot$ are allowed, coNP-complete if at least conjunctions and at most conjunctions plus the constant $\bot$ are allowed, and trivial otherwise, *i.e.*, for all other sets of Boolean operators, every modal formula (concept description) is satisfiable. We will put this classification into the context of the above listed results for $\mathcal{ALC}$ fragments.

The tool used in [16] for classifying the infinitely many satisfiability problems was Post's lattice [23], which consists of all sets of Boolean functions closed under superposition. These sets directly correspond to all sets of Boolean operators closed under nesting. Similar classifications have been achieved for satisfiability for classical propositional logic [19], Linear Temporal Logic [7], hybrid logic [20], and for constraint satisfaction problems [25, 27].

In this paper, we classify the concept satisfiability problems with respect to theories for $\mathcal{ALC}$ fragments obtained by arbitrary sets of Boolean operators. We will separate tractable and intractable cases, showing that these problems are

- EXPTIME-hard whenever we allow at least conjunction, disjunction or all self-dual operators, where a Boolean function is called self-dual if negating all its arguments negates its value,
- PSPACE-hard whenever we allow at least negation or both constants $\bot, \top$,
- coNP-hard whenever we allow at least the constant $\bot$,
- trivial, which means that all instances are satisfiable, in all other cases.

We will also put these results into the context of the above listed results for $\mathcal{ALC}$ fragments. This is work in progress which we plan to extend by corresponding upper bounds, restricted use of $\exists, \forall$, and terminological restrictions to TBoxes such as acyclicity and atomic left-hand sides of axioms. Furthermore, not all results carry over straightforwardly to other reasoning problems because some of the standard reductions use Boolean operators that are not available in every fragment.

2 Preliminaries

Description Logic. We use the standard syntax and semantics of $\mathcal{ALC}$ with the Boolean operators $\sqcap$, $\sqcup$, $\neg$, $\top$, $\bot$ replaced by arbitrary operators o that correspond to Boolean functions f_o of arbitrary arity. Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be sets of atomic concepts, roles and individuals. Then the set of *concept descriptions*, for short *concepts*, is defined by

$$C := A \mid o(C, \dots, C) \mid \exists R.C \mid \forall R.C,$$

where $A \in \mathsf{N_C}$, $R \in \mathsf{N_R}$, and o is a Boolean operator. A *general concept inclusion (GCI)* is an axiom of the form $C \sqsubseteq D$ where C, D are concepts. We use "$C \equiv D$" as the usual syntactic sugar for "$C \sqsubseteq D$ and $D \sqsubseteq C$". A *TBox* is a finite set of GCIs without restrictions. An *ABox* is a finite set of axioms of the form $C(x)$ or $R(x, y)$, where C is a concept, $R \in \mathsf{N_R}$ and $x, y \in \mathsf{N_I}$. An *ontology* is the union of a TBox and an ABox. This simplified view suffices for our purposes.

An *interpretation* is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where $\Delta^\mathcal{I}$ is a nonempty set and $\cdot^\mathcal{I}$ is a mapping from $\mathsf{N_C}$ to $\mathfrak{P}(\Delta^\mathcal{I})$, from $\mathsf{N_R}$ to $\mathfrak{P}(\Delta^\mathcal{I} \times \Delta^\mathcal{I})$ and from $\mathsf{N_I}$ to $\Delta^\mathcal{I}$ that is extended to arbitrary concepts as follows:

$$\begin{aligned}
o(C_1, \dots, C_n)^\mathcal{I} &= \{x \in \Delta^\mathcal{I} \mid f_o(\|x \in C_1^\mathcal{I}\|, \dots, \|x \in C_n^\mathcal{I}\|) = \top\}, \\
&\qquad \text{where } \|x \in C_1^\mathcal{I}\| = \top \text{ if } x \in C_1^\mathcal{I} \text{ and } \|x \in C_1^\mathcal{I}\| = \bot \text{ if } x \notin C_1^\mathcal{I}, \\
\exists R.C^\mathcal{I} &= \{x \in \Delta^\mathcal{I} \mid \{y \in C^\mathcal{I} \mid (x, y) \in R^\mathcal{I}\} \neq \emptyset\}, \\
\forall R.C^\mathcal{I} &= \{x \in \Delta^\mathcal{I} \mid \{y \in C^\mathcal{I} \mid (x, y) \notin R^\mathcal{I}\} = \emptyset\}.
\end{aligned}$$

An interpretation $\mathcal{I}$ *satisfies* the axiom $C \sqsubseteq D$, written $\mathcal{I} \models C \sqsubseteq D$, if $C^\mathcal{I} \subseteq D^\mathcal{I}$. Furthermore, $\mathcal{I}$ satisfies $C(x)$ or $R(x, y)$ if $x^\mathcal{I} \in C^\mathcal{I}$ or $(x^\mathcal{I}, y^\mathcal{I}) \in R^\mathcal{I}$. An interpretation $\mathcal{I}$ satisfies a TBox (ABox, ontology) if it satisfies every axiom therein. It is then called a *model* of this set of axioms.

Let B be a finite set of Boolean operators and use $\mathsf{Con}(B)$ and $\mathsf{Ax}(B)$ to denote the set of all concepts and axioms using only operators in B. The following decision problems are of interest for this paper.

Concept satisfiability CSAT(B):
Given a concept $C \in \mathsf{Con}(B)$, is there an interpretation $\mathcal{I}$ s.t. $C^\mathcal{I} \neq \emptyset$?

TBox satisfiability TSAT(B):
Given a TBox $\mathcal{T} \subseteq \mathsf{Ax}(B)$, is there an interpretation $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{T}$?

TBox-concept satisfiability TCSAT(B):
Given $\mathcal{T} \subseteq \mathsf{Ax}(B)$ and $C \in \mathsf{Con}(B)$, is there an $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{T}$ and $C^\mathcal{I} \neq \emptyset$?

Ontology satisfiability OSAT(B):
Given an ontology $\mathcal{O} \subseteq \mathsf{Ax}(B)$, is there an interpretation $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{O}$?

Ontology-concept satisfiability OCSAT(B):
Given $\mathcal{O} \subseteq \mathsf{Ax}(B)$ and $C \in \mathsf{Con}(B)$, is there an $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{O}$ and $C^{\mathcal{I}} \neq \emptyset$?

These problems are interreducible independently of B in the following way:

$$\mathrm{CSAT}(B) \leq_{\mathrm{m}}^{\log} \mathrm{OSAT}(B)$$
$$\mathrm{TSAT}(B) \leq_{\mathrm{m}}^{\log} \mathrm{TCSAT}(B) \leq_{\mathrm{m}}^{\log} \mathrm{OSAT}(B) \equiv_{\mathrm{m}}^{\log} \mathrm{OCSAT}(B)$$

The reasons are: a concept C is satisfiable iff the ontology $\{a : C\}$ is satisfiable, for some individual a; a terminology $\mathcal{T}$ is satisfiable iff a fresh atomic concept A is satisfiable w.r.t. $\mathcal{T}$; C is satisfiable w.r.t. $\mathcal{T}$ iff $\mathcal{T} \cup \{a : C\}$ is satisfiable, for a fresh individual a.

Complexity Theory. We assume familiarity with the standard notions of complexity theory as, e. g., defined in [22]. In particular, we will make use of the classes P, NP, coNP, PSPACE, and EXPTIME, as well as logspace reductions $\leq_{\mathrm{m}}^{\log}$.

Boolean operators. This study aims at being complete with respect to Boolean operators, which correspond to Boolean functions. A set of Boolean functions is called a *clone* if it is closed under superpositions of functions, *i.e.*, nesting of operators. The lattice of all clones has been established in [23], see [8] for a more succinct but complete presentation. Via the inclusion structure, lower and upper complexity bounds carry over to higher and lower clones. We will therefore only state our results for minimal and maximal clones.

Given a finite set B of functions, the smallest clone containing B is denoted by $[B]$. The set B is called a *base* of $[B]$, but $[B]$ often has other bases as well. On the operator side, $[B]$ consists of all operators obtained by nesting operators in B into each other. For example, nesting of binary conjunction yields conjunctions of arbitrary arity. The table below lists all clones that we will refer to, using the following definitions. A Boolean function f is called *self-dual* if $f(\overline{x_1}, \ldots, \overline{x_n}) = \overline{f(x_1, \ldots, x_n)}$, *c-reproducing* if $f(c, \ldots, c) = c$, and *c-separating* if there is an $1 \leq i \leq n$ s.t. for each $(b_1, \ldots, b_n) \in f^{-1}(c)$ $b_i = c$ for $c \in \{\top, \bot\}$. The symbol $\oplus$ denotes the binary exclusive or.

Clone	Description	Base
BF	all Boolean functions	$\{\wedge, \neg\}$
M	All monotone functions	$\{\wedge, \vee, \bot, \top\}$
S_{11}	$\top$-separating, monotone function	$\{x \wedge (y \vee z), \bot\}$
D	self-dual functions	$\{(x \wedge \overline{y}) \vee (x \wedge \overline{z}) \vee (\overline{y} \wedge \overline{z})\}$
E	conjunctions and constants	$\{\wedge, \bot, \top\}$
E_0	conjunctions and $\bot$	$\{\wedge, \bot\}$
V_0	disjunctions and $\bot$	$\{\vee, \bot\}$
R_1	$\top$-reproducing functions	$\{\vee, x \oplus y \oplus \top\}$
R_0	$\bot$-reproducing functions	$\{\wedge, \oplus\}$
N_2	negation	$\{\neg\}$
I	identity functions and constants	$\{\mathrm{id}, \bot, \top\}$
I_0	identity functions and $\bot$	$\{\mathrm{id}, \bot\}$

The following lemma will help restrict the length of concepts in some of our reductions. It shows that for certain sets B, there are always short concepts representing the functions $\wedge$, $\vee$, or $\neg$, respectively. Points (2) and (3) follow directly from the proofs in [19], Point (1) is Lemma 1.4.5 from [27].

Lemma 1. *Let B be a finite set of Boolean functions.*

1. *If $\mathsf{V} \subseteq [B] \subseteq \mathsf{M}$ ($\mathsf{E} \subseteq [B] \subseteq \mathsf{M}$, resp.), then there exists a B-formula $f(x,y)$ such that f represents $x \vee y$ ($x \wedge y$, resp.) and each of the variables x and y occurs exactly once in $f(x,y)$.*
2. *If $[B] = \mathsf{BF}$, then there are B-formulae $f(x,y)$ and $g(x,y)$ such that f represents $x \vee y$, g represents $x \wedge y$, and both variables occur in each of these formulae exactly once.*
3. *If $\mathsf{N} \subseteq [B]$, then there is a B-formula $f(x)$ such that f represents $\neg x$ and the variable x occurs in f only once.*

Auxiliary results. The following lemmata contain technical results that will be useful to formulate our main results. We use $\star\mathrm{SAT}(B)$ to speak about any of the four satisfiability problems $\mathrm{TSAT}, \mathrm{TCSAT}, \mathrm{OSAT}$ and OCSAT introduced above.

Lemma 2. *Let B be a finite set of Boolean functions. If $\mathsf{N}_2 \subseteq [B]$, then it holds that $\star\mathrm{SAT}(B) \equiv_{\mathrm{m}}^{\log} \star\mathrm{SAT}(B \cup \{\top, \bot\})$.*

Proof. It is easy to observe that the concepts $\top$ and $\bot$ can be simulated by fresh atomic concepts T and B, using the axioms $\neg T \sqsubseteq T$ and $B \sqsubseteq \neg B$. □

Lemma 3. *Let B be a finite set of Boolean functions. Then it holds that $\mathrm{TCSAT}(B) \leq_{\mathrm{m}}^{\log} \mathrm{TSAT}(B \cup \{\top\})$.*

Proof. It can be easily shown that $\langle C, \mathcal{T} \rangle \in \mathrm{TCSAT}(B)$ iff $\langle \mathcal{T} \cup \{\top \sqsubseteq \exists R.C\} \rangle \in \mathrm{TSAT}(B \cup \{\top\})$, where R is a fresh relational symbol. For "$\Rightarrow$" observe that for the satisfying interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ there must be a world w' where C holds and then from every world $w \in \Delta^{\mathcal{I}}$ there can be an R-edge from w to w' to satisfy $\mathcal{T} \cup \{\top \sqsubseteq \exists R.C\}$. For "$\Leftarrow$" note that for a satisfying interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ all axioms in $\mathcal{T} \cup \{\top \sqsubseteq \exists R.C\}$ are satisfied. In particular the axiom $\top \sqsubseteq \exists R.C$. Hence there must be at least one world w' s.t. $w' \models C$. Thus $\mathcal{I} \models \mathcal{T}$ and $C^{\mathcal{I}} \supseteq \{w'\} \neq \emptyset$. □

Furthermore, we observe that, for each set B of Boolean functions with $\top, \bot \in [B]$, we can simulate the negation of an atomic concept using a fresh atomic concept A and role R_A: if we add the axioms $A \equiv \exists R_A.\top$ and $A' \equiv \forall R_A.\bot$ to the given terminology $\mathcal{T}$, then each model of $\mathcal{T}$ has to interpret A' as the complement of A.

3 Complexity results for CSAT

The following classification of concept satisfiability has been obtained in [16].

Theorem 4 ([16]). *Let B be a finite set of Boolean functions.*

1. *If* $\mathsf{S}_{11} \subseteq [B]$*, then* CSAT$(B)$ *is* PSPACE*-complete.*
2. *If* $[B] \in \{\mathsf{E}, \mathsf{E}_0\}$*, then* CSAT$(B)$ *is* coNP*-complete.*
3. *If* $[B] \subseteq \mathsf{R}_1$*, then* CSAT$(B)$ *is trivial.*
4. *Otherwise* CSAT$(B) \in$ P*.*

Part (1) is in contrast with the coNP-completeness of $\mathcal{ALU}$ satisfiability [26] because the operators in $\mathcal{ALU}$ can express the canonical base of S_{11}. The difference is caused by the fact that $\mathcal{ALU}$ allows only *unqualified* existential restrictions. Part (2) generalises the coNP-completeness of $\mathcal{ALE}$ satisfiability, where hardness is proven in [12] without using atomic negation. It is in contrast with the tractability of $\mathcal{AL}$ satisfiability [13], again because of the unqualified restrictions. Part (3) generalises the known fact that every $\mathcal{EL}$, $\mathcal{FL}_0$, and $\mathcal{FL}^-$ concept is satisfiable. The results for logics in the DL-Lite family cannot be put into this context because DL-Lite quantifiers are unqualified.

4 Complexity Results for TSAT, TCSAT, OSAT, OCSAT

In this section we will completely classify the above mentioned satisfiability problems for their tractability with respect to sub-Boolean fragments and put them into context with existing results for fragments of $\mathcal{ALC}$.

Main results. Due to the interreducibilities stated in Section 2, it suffices to show lower bounds for TSAT and upper bounds for OCSAT.

Theorem 5. *Let B be a finite set of Boolean functions.*

1. *If* $\wedge \in B$ *or* $\vee \in B$*, then* TCSAT(B) *is* EXPTIME*-hard.*
   If also $\top \in B$*, then even* TSAT(B) *is* EXPTIME*-hard.*
2. *If all functions in B are self-dual, then* TSAT(B) *is* EXPTIME*-hard.*
3. *If* $\neg \in B$ *or* $\{\top, \bot\} \subseteq B$*, then* TSAT$(B)$ *is* PSPACE*-hard.*
4. *If all functions in B are $\bot$-reproducing, then* TSAT(B) *is trivial.*
5. *If* $\bot \in B$*, then* TCSAT(B) *is* coNP*-hard.*
6. *If all functions in B are $\top$-reproducing, then* OCSAT(B) *is trivial.*

Proof. Parts 1.–6. are formulated as Lemmas 11, 12, 13, 9, 10, 14, and are proven below. The second part of (1.) follows from Lemma 11 in combination with Lemma 3. □

In order to generalize these results, we need to prove the following lemma. It states the base independence that will lead to the more general results in Theorem 7 and Theorem 8.

Lemma 6. *Let B_1, B_2 be two sets of Boolean functions s.t. $[B_1] = [B_2]$. Then* $\star$SAT$(B_1) \leq_{\mathrm{m}}^{\log} \star$SAT$(B_2)$*.*

Proof. According to [16, Theorem 3.6], we translate for any given instance each Boolean formula (hence each side of an axiom) into a Boolean circuit over the basis B_1. This circuit can be easily transformed into a circuit over the basis B_2. This new circuit will be expressed by several new axioms that are constructed in the style of the formulae in [16]:

- For input gates g, we add the axiom $g \equiv x_i$.
- If g is a gate computing the Boolean function ϕ and $h_1, \ldots, h_n$ are the respective predecessor gates in this circuit, we add the axiom $g \equiv \phi(h_1, \ldots, h_n)$.
- For $\exists R$-gates g, we add the axiom $g \equiv \exists R.h$.
- Analogously for $\forall R$-gates.

For each axiom $A \sqsubseteq B$, let g^A_{out} and g^B_{out} be the output gates of the appropriate circuits. Then we need to add one new axiom $g^A_{out} \sqsubseteq g^B_{out}$ to ensure the axiomatic property of $A \sqsubseteq B$. If the translated formula ψ is a given concept expression (relevant for the problems $\mathrm{TCSAT}, \mathrm{OCSAT}$), the translated concept is mapped to the respective out-gate g^{ψ}_{out}.

This reduction is computable in logarithmic space and its correctness can be shown in the same way as in the proof of Theorem 3.6 in [16]. □

As a consequence of Theorem 5 in combination with Lemma 6 , we obtain the following two corollaries that generalise the results to arbitrary bases for all four satisfiability problems.

Corollary 7. *Let B be a finite set of Boolean functions and $\star\mathrm{SAT}'$ one of the problems* TCSAT, OSAT *and* OCSAT.

1. *If $\mathsf{E}_0 \subseteq [B]$ or $\mathsf{V}_0 \subseteq [B]$, and $[B] \subseteq \mathsf{M}$, then $\star\mathrm{SAT}'(B)$ is* EXPTIME*-hard.*
2. *If $[B] = \mathsf{D}$ or $[B] = \mathsf{BF}$, then $\star\mathrm{SAT}'(B)$ is* EXPTIME*-hard.*
3. *If $\mathsf{N}_2 \subseteq [B]$ or $\mathsf{I} \subseteq [B]$, then $\star\mathrm{SAT}'(B)$ is* PSPACE*-hard.*
4. *If $[B] = \mathsf{I}_0$, then $\star\mathrm{SAT}'(B)$ is* coNP*-hard.*
5. *If $[B] \subseteq \mathsf{R}_1$, then $\star\mathrm{SAT}'(B)$ is trivial.*

Corollary 8. *Let B be a finite set of Boolean functions.*

1. *If $\mathsf{E} \subseteq [B]$ or $\mathsf{V} \subseteq [B]$, and $[B] \subseteq \mathsf{M}$, then $\mathrm{TSAT}(B)$ is* EXPTIME*-hard.*
2. *If $[B] = \mathsf{D}$ or $[B] = \mathsf{BF}$, then $\mathrm{TSAT}(B)$ is* EXPTIME*-hard.*
3. *If $\mathsf{N}_2 \subseteq [B]$ or $\mathsf{I} \subseteq [B]$, then $\mathrm{TSAT}(B)$ is* PSPACE*-hard.*
4. *If $[B] \subseteq \mathsf{R}_0$, then $\mathrm{TSAT}(B)$ is trivial.*
5. *If $[B] \subseteq \mathsf{R}_1$, then $\mathrm{TSAT}(B)$ is trivial.*

Part (1) generalises the EXPTIME-hardness of subsumption for $\mathcal{FL}_0$ and $\mathcal{AL}$ with respect to GCIs [15, 11, 17]. It is in contrast to the tractability of subsumption with respect to GCIs in $\mathcal{EL}$ because our result does not separate the two types of restriction, because $\mathcal{EL}$ has only existential restriction, and our results do not (yet) consider existential, resp., universal restrictions separately. This undermines the observation that, for negation-free fragments, the choice of the quantifier affects tractability and not the choice between conjunction and disjunction. Again, DL-Lite cannot be put into this context because of the unqualified restrictions.

Parts (2)–(4) (resp. (2) and (3) for Theorem 8) show that satisfiability with respect to theories is already intractable for even smaller sets of Boolean operators. One reason is that sets of axioms already contain limited forms of implication and conjunction. This also causes the results of this analysis to differ from similar analyses for related logics in that hardness already holds for bases of clones that are comparatively low in Post's lattice.

Due to Post's lattice, our analysis is complete for dividing the fragments into tractable and intractable cases.

Proofs of the main results.

Lemma 9. *Let B be a finite set of Boolean functions s.t. B contains only $\top$-reproducing functions. Then* $\mathrm{OCSAT}(B)$ *is trivial.*

Proof. According to Post's lattice, every B that does not fall under Theorem 5 (1)–(4)+(6) contains only $\top$-reproducing functions. Hence the following interpretation satisfies any instance $(\mathcal{O}, C)$: $\mathcal{I} = (\{w\}, \cdot^{\mathcal{I}})$ s.t. $A^{\mathcal{I}} = \{w\}$ for each atomic concept A, $r^{\mathcal{I}} = \{(w, w)\}$ for each role r, and $a^{\mathcal{I}} = w$ for each individual a. It then holds trivially that $\mathcal{I} \models \mathcal{O}$ and $C^{\mathcal{I}} = \{w\} \neq \emptyset$. □

Lemma 10. *Let B be a finite set of Boolean functions s.t. B contains only $\bot$-reproducing functions. Then* $\mathrm{TSAT}(B)$ *is trivial.*

Proof. The intepretation $\mathcal{I} = (\{w\}, \cdot^{\mathcal{I}})$ with $A^{\mathcal{I}} = \emptyset$ for each atomic concept A, and $r^{\mathcal{I}} = \{(w, w)\}$ for each role r satisfies any instance $\mathcal{T}$ for $\mathrm{TSAT}(B)$, where B contains only $\bot$-reproducing functions. This follows from the observation that for each axiom $A \sqsubseteq B$ in $\mathcal{T}$ both sides are always falsified by $\mathcal{I}$ (because every atomic concept is falsified, and we only have $\bot$-reproducing operators as connectives). This can be shown by an easy induction on the concept structure. Please note that we need to construct a looping node concerning the transition relations due to the fact that we need to falsify axioms with $\forall r.\bot$ on the left side for some relation r. If we set $r^{\mathcal{I}} = \emptyset$ then this expression would be satisfied and would contradict our argumentation for the axiom $\forall r.\bot \sqsubseteq \bot$. Moreover this construction cannot fulfill wrongly the left side of an axiom because of the absence of $\top$ and as no atomic concept has instances with w. □

Lemma 11. *Let B be a finite set of Boolean functions with $\wedge \in B$, or $\vee \in B$. Then* $\mathrm{TCSAT}(B)$ *is* EXPTIME*-hard. If all self-dual functions can be expressed in B, then* $\mathrm{TSAT}(B)$ *is* EXPTIME*-hard.*

Proof. The cases $\wedge \in B$ and $\vee \in B$ follow from [15]. The remaining case for the self-dual functions follows from Lemmas 1 and 2, as all self-dual functions in combination with the constants $\top, \bot$ (to which we have access as $\neg$ is self-dual) can express any arbitrary Boolean function. □

Lemma 12. *Let B be a finite set of Boolean functions s.t. $\{\bot, \top\} \subseteq B$. Then* $\mathrm{TSAT}(B)$ *is* PSPACE*-hard.*

Proof. To prove PSPACE-hardness, we state a $\leq_{\mathrm{cd}}$-reduction from QBF-3-SAT to TSAT(B) and only allow $\bot$ and $\top$ as available functions in B. Let $\varphi \equiv \partial_1 x_1 \partial_2 x_2 \cdots \partial_n x_n (C_1 \wedge \cdots \wedge C_m)$ be a quantified Boolean formula and $\partial_i \in \{\exists, \forall\}$. In the following we construct a TBox $\mathcal{T} \subseteq \mathsf{Ax}(B)$ s.t. $\varphi \equiv \top$ if and only if $\mathcal{T} \in \mathrm{TSAT}(B)$, where B consists only of $\top$ and $\bot$.

We are first adding the following axioms to the TBox $\mathcal{T}$ using atomic concepts $d_0, \ldots, d_n, x_1, \ldots, x_n, x'_1, \ldots, x'_n$ and roles $R_r, R_1, \ldots, R_n, S, R_{x_1}, \ldots, R_{x_n}$, $R_{d_1}, \ldots, R_{d_n}$, $R_{C_1}, \ldots, R_{C_m}, P_{11}, P_{21}, P_{31}, \ldots, P_{1m}, P_{2m}, P_{3m}$. The atomic concepts d_i stand for levels, x_i and x'_i for assigning truth values to the variables.

Initial starting point:

$$\{\top \sqsubseteq \exists S.d_0\} \tag{1}$$

x_i is the negation of x'_i:

$$\{x_i \equiv \exists R_{x_i}.\top \mid 1 \leq i \leq n\} \cup \{x'_i \equiv \forall R_{x_i}.\bot \mid 1 \leq i \leq n\} \tag{2}$$

in each level d_i we have R_{i+1}-successors where x_{i+1} and x'_{i+1} hold:

$$\{d_i \sqsubseteq \exists R_{i+1}.x_{i+1} \mid 0 \leq i < n\} \cup \{d_i \sqsubseteq \exists R_{i+1}.x'_{i+1} \mid 0 \leq i < n\} \tag{3}$$

the levels d_i are disjoint and we have succeeding levels:

$$\{d_i \sqsubseteq \forall R_{i+1}.d_{i+1} \mid 0 \leq i < n\} \cup \{d_i \sqsubseteq \exists R_{d_i}.\top, d_j \sqsubseteq \forall R_{d_i}.\bot \mid 0 \leq i < j \leq n\} \tag{4}$$

x_i and x'_i carry over:

$$\{x_i \sqsubseteq \forall R_j.x_i \mid 1 \leq i < j \leq n\} \cup \{x'_i \sqsubseteq \forall R_j.x'_i \mid 1 \leq i < j \leq n\} \tag{5}$$

Now $\mathcal{T}$ is consistent, and each of its models contains a tree-like substructure similar to the one depicted in Figure 2. The *root* of this substructure is an instance of d_0. The individuals at depth n counting from the root are called *leaves*.

Please note that each individual in $\Delta^{\mathcal{I}}$ is an instance of either x_i or x'_i because of axiom (2). In particular, this holds for the leaves. Furthermore, this enforcement does not contradict the level-based labeling of the x_i—*e.g.*, the atomic concepts x_i and x'_i "labeled in d_0" are *not* carried forward to the next levels because axiom (5) states this carry only if $j > i$.

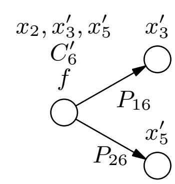

Fig. 1. clause $C_6 \equiv \overline{x}_2 \vee x_3 \vee x_5$

In the remaining part, we need to ensure the following, where C_j is an arbitrary clause in φ. Each leaf w is an instance of the atomic concept C_j if and only if the combination of the x_i-values in w satisfies the clause C_j. In order to achieve this, we again use two complementary atomic propositions C_j and C'_j. The C'_j must be enforced in all leaves where *all* literals of C_j are set to false. For a literal $\ell \in \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$, use $\widetilde{\ell}$ to denote the atomic concept x_i if $\ell = \overline{x}_i$ and x'_i if $\ell = x_i$. The correct labeling of the leaves by the C_j and C'_j is ensured by adding the following axioms to $\mathcal{T}$, which enforce substructures as depicted for the example in Figure 1:

$$\Big\{\widetilde{l}_{1j} \sqsubseteq \exists P_{1j}.\top,\ \ \widetilde{l}_{2j} \sqsubseteq \forall P_{1j}.\widetilde{l}_{2j},\ \ \exists P_{1j}.\widetilde{l}_{2j} \sqsubseteq \exists P_{2j}.\top,$$
$$\widetilde{l}_{3j} \sqsubseteq \forall P_{2j}.\widetilde{l}_{3j},\ \ \exists P_{2j}.\widetilde{l}_{3j} \sqsubseteq C'_j,\ \Big|\ C_j = l_{1j} \vee l_{2j} \vee l_{3j} \text{ in } \varphi\Big\} \ \cup \tag{6}$$
$$\big\{C'_j \sqsubseteq f \ \mid 1 \leq j \leq m\big\} \cup \tag{7}$$
$$\{f \sqsubseteq \exists F.\top,\ \ f' \sqsubseteq \forall F.\bot\} \cup \tag{8}$$
$$\big\{C_j \equiv \exists R_{C_j}.\top,\ \ C'_j \equiv \forall R_{C_j}.\bot \mid 1 \leq j \leq m\big\} \tag{9}$$

Finally we need to ensure that all concepts C_j are true in the leaves depending on the quantifications $\reflectbox{$\partial$}_1 x_1 \reflectbox{$\partial$}_2 x_2 \cdots \reflectbox{∂}_n x_n$. For this purpose, we add the following axioms to the TBox $\mathcal{T}$ which ensure that, starting at the root, we run through each variable level of the tree as required by the quantification in φ, and reach only leaves that are no instances of f, *i.e.*, that are instances of f':

$$\{d_0 \sqsubseteq \reflectbox{∂}_1 R_1.\reflectbox{∂}_2 R_2.\cdots \reflectbox{∂}_n R_n.f'\} \tag{10}$$

Claim. $\varphi \equiv \top$ iff $\mathcal{T} \in \mathrm{TSAT}(\{\top, \bot\})$.

Proof. "$\Leftarrow$": Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation s.t. $\mathcal{I} \models \mathcal{T}$. Due to axiom (1), there exists an individual w_0 that is an instance of d_0. Because of axioms (3) and (4), there are at least two different R_1-successors of d_0, one being an instance of x_1 and the other of x'_1 (axiom (5) in combination with axiom (4) ensure that these successors are fresh individuals). *Every* other R_1-successor is an instance of either x_1 or x'_1, due to axiom (2). Other possible R_j-edges for $2 \leq j \leq n$ will not affect our argumentation as we will see in the following.

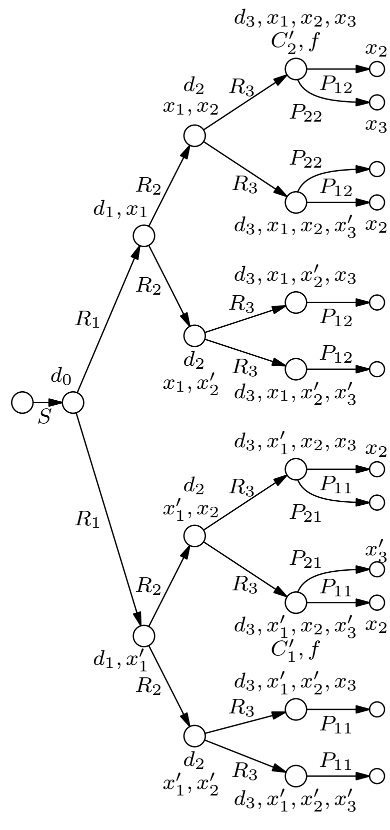

Fig. 2. Essential part of the interpretation for the qBf $\varphi = \exists x_1 \forall x_2 \exists x_3 (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$.

Repeated application of axioms (3) and (4) shows that this structure becomes a complete binary tree of depth n with (at least) 2^n leaves. Each leaf represents one of all possible Boolean combinations of x_i and x'_i for $1 \leq i \leq n$. Due to axioms (3) and (4), every possible combination does occur. In addition, axiom (9) and (7) ensure the following: each leaf is an instance of either C_j or C'_j, for each $1 \leq j \leq m$; if a leaf is an instance of at least one such C_j, it is also an instance of f.

Axiom (10) allows us to conclude that all relevant leaves that represent the assignments $\theta_i\colon \{x_1, \dots, x_n\} \to \{0,1\}$ for which $\theta_i \models C_1 \wedge \dots \wedge C_m$ must hold, are instances of the proposition f'. Additional R_j-edges, as mentioned above, do not contradict the argumentation. Hence every relevant leaf must be an instance of every C_j because otherwise it were an instance of C'_j and thus of f'. Therefore, at least one literal in each clause is labeled and thereby satisfied. Hence $\varphi \equiv \top$.

Note that only those leaves that correspond to an assignment satisfying C_j can be instances of C_j. To clarify this fact, consider a clause $C_j = l_{1j} \vee l_{2j} \vee l_{3j}$ that is not satisfied by some assignment $\theta\colon \{x_1, \dots, x_n\} \to \{\top, \bot\}$, and some leaf w is (erroneously) an instance of C_j. As $\theta \not\models C_j$, it holds that $\theta \not\models l_{ij}$ for $1 \leq i \leq 3$. Thus l'_{ij} must be labeled in w in order for axiom (2) to be satisfied. Now axiom (6) enforces R_{1j}- and R_{2j}-edges to successors satisfying $\widetilde{l}_{2j}$ and $\widetilde{l}_{3j}$. Finally, these propositions and transitions lead to w being an instance of C'_j. This is not possible because C_j and C'_j are disjoint due to axiom (9).

"$\Rightarrow$": Let n be the number of variables in φ. In the following we show by induction on n: if $\varphi = \exists x_1 \forall x_2 \cdots \reflectbox{$\exists$} x_n (C_1 \wedge \dots \wedge C_m) \equiv \top$, then $\mathcal{T} \in \mathrm{TSAT}(\{\top, \bot\})$.

Induction basis. $n = 1$. W.l.o.g., we assume that φ starts with $\exists$, *i.e.*, $\varphi = \exists x_1 (C_1 \wedge \dots \wedge C_m) \equiv \top$, and we assume that each C_i contains the positive literal x_1.

We construct a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where we set $\Delta^{\mathcal{I}} = \{w_0, w_1, w_2, w_3\}$, $(S)^{\mathcal{I}} = \{(w_0, w_1), (w_1, w_1), (w_2, w_1), (w_3, w_1)\}$, $(R_1)^{\mathcal{I}} = \{(w_1, w_2), (w_1, w_3)\}$, $(R_{x_1})^{\mathcal{I}} = \{(w_0, w_0), (w_1, w_1), (w_2, w_2)\}$, $(d_0)^{\mathcal{I}} = \{w_1\}$, $(d_1)^{\mathcal{I}} = \{w_2, w_3\}$, $(x_1)^{\mathcal{I}} = \{w_0, w_1, w_2\}$, $(x'_1)^{\mathcal{I}} = \{w_3\}$. Then $(C_j)^{\mathcal{I}} = \{w_0, w_1, w_2\}$, $(C'_j)^{\mathcal{I}} = \{w_3\}$ and $(R_{C_j})^{\mathcal{I}} = \{(w_2, w_2)\}$ for all $1 \leq j \leq m$. Finally $(f')^{\mathcal{I}} = \{w_2\}$ (the remaining labels are irrelevant). From this it can be easily verified that $\mathcal{I} \models \mathcal{T}$.

Induction step. Assume it holds for $n \geq 1$. In the following we will show that the proposition holds for $\varphi = \exists x_1 \forall x_2 \cdots \reflectbox{$\exists$} x_{n+1} F \equiv \top$ with $F = (C_1 \wedge \dots \wedge C_m)$. If φ starts with $\forall$, the argumentation is analogous. Let $F[x_1/\top]$ (or $F[x_1/\bot]$) denote the matrix F of φ with every occurrence of x_1 replaced by $\top$ (or $\bot$). Since $\varphi \equiv \top$, we have that $\chi_1 = \forall x_2 \exists x_3 \cdots \reflectbox{$\exists$} x_{n+1} F[x_1/\top] \equiv \top$ or $\chi_2 = \forall x_2 \exists x_3 \cdots \reflectbox{$\exists$} x_{n+1} F[x_1/\bot] \equiv \top$.

Assume $\chi_1 = \forall x_2 \exists x_3 \cdots \reflectbox{$\exists$} x_{n+1} F[x_1/\top] \equiv \top$. Let $\mathcal{T}_{\chi_1}$ be the consistent terminology that is constructed out of χ_1 (this follows from our induction hypothesis). For each variable x_i let $\mathcal{C}_i = \{C_j \mid x_i \in C_j, 1 \leq j \leq m\}$ and $\mathcal{C}'_i = \{C_j \mid \overline{x}_i \in C_j, 1 \leq j \leq m\}$ be the set of clauses that include the literal x_i resp. $\overline{x}_i$. Let $\theta_1, \theta_2, \dots, \theta_k$ with $\theta_i\colon \{x_2, x_3, \dots, x_{n+1}\} \to \{0,1\}$ be the satisfying assignments generated by $\exists x_2 \forall x_3 \cdots \reflectbox{$\exists$} x_{n+1}$. As $\mathcal{T}_{\chi_1}$ is consistent, let $\mathcal{I}_{\chi_1} = (\Delta^{\mathcal{I}_{\chi_1}}, \cdot^{\mathcal{I}_{\chi_1}})$ be an interpretation s.t. $\mathcal{I}_{\chi_1} \models \mathcal{T}_{\chi_1}$. Hence it satisfies every axiom in $\mathcal{T}_{\chi_1}$ (that has the form of above) and in particular axiom (1). Thus there is an individual $w_0 \in \Delta^{\mathcal{I}_{\chi_1}}$ s.t. $w_0 \in (d_0)^{\mathcal{I}_{\chi_1}}$ and therefore a binary tree is generated starting in w_0 (with the same argumentation as before). As that tree is defined over the variables $x_2, x_3, \dots, x_{n+1}$, all leaves addressed by axiom (10) include $C_1, C_2, \dots, C_m, f'$, and for the transition relation R_2 it holds $(R_2)^{\mathcal{I}_{\chi_1}} \supseteq \{(w_0, w_2), (w_0, w'_2)\}$ with $(x_2)^{\mathcal{I}_{\chi_1}} \ni w_2$, $(x'_2)^{\mathcal{I}_{\chi_1}} \ni w'_2$.

In the next steps we will construct an interpretation $\mathcal{I}_\varphi$ s.t. $\mathcal{I}_\varphi \models \mathcal{T}_\varphi$. Therefore we start with the previous interpretation $\mathcal{I}_{\chi_1}$ and modify it into $\mathcal{I}_\varphi$ in the following steps as visualized in Figure 3:

Set $\Delta^{\mathcal{I}_\varphi} := \Delta^{\mathcal{I}_{\chi_1}}$. Now Define $(R_1)^{\mathcal{I}_\varphi} = \{(w_0, w_1)\}$ with w_1 a new individual added to $\Delta^{\mathcal{I}_\varphi}$. This inserts the first branch at the top for variable x_1, and hence we add w_1 to $(x_1)^{\mathcal{I}}$. Now replace $(w_0, w_2), (w_0, w_2')$ from $(R_2)^{\mathcal{I}_\varphi}$ with $(w_1, w_2), (w_1, w_2')$ to mount the binary tree in $\mathcal{I}_{\chi_1}$ below the node $w_1 \in \Delta^{\mathcal{I}_\varphi}$ that represents choosing x_1. Also add w_1 to the set $(d_1)^{\mathcal{I}}$. Now let $\mathcal{T} \subset \Delta^{\mathcal{I}_{\chi_1}}$ be the set of individuals that form the binary tree in $\mathcal{I}_{\chi_1}$ starting in d_0. For each individual in $\mathcal{T} \setminus \{w_0\}$ we need to increment the value of the d_is by one. Set $\widetilde{\mathcal{T}} := \{\widetilde{w} \mid w \in \mathcal{T} \setminus \{w_0, w_s\}$ plus the changes on the d_is as before$\}$. Add each $w \in \widetilde{\mathcal{T}}$ to $\Delta^{\mathcal{I}_\varphi}$ and for each relation $R \in \mathsf{N_R} \setminus \{S\}$ and each wRw' with $w, w' \in \mathcal{T}$ add $(\widetilde{w}, \widetilde{w}')$ to $R^{\mathcal{I}_\varphi}$. For each $w \in \widetilde{\mathcal{T}}$ and each proposition p with $w \in (p)^{\mathcal{I}_{\chi_1}}$ also add $\widetilde{w}$. Now we have a complete copy (all nodes and edges) of the binary assignment tree constructed and added to our interpretation $\mathcal{I}_\varphi$. In the next step we mount the previously added tree below w_0. Here we add a new individual w_1' to $\Delta^{\mathcal{I}}$, $w_1' \in (x_1')^{\mathcal{I}}$, $w_1' \in (d_1)^{\mathcal{I}}$ and $(w_0, w_1') \in (R_1)^{\mathcal{I}}$. Now we add with $(w_1', \widetilde{w}_2), (w_1', \widetilde{w}_2') \in (R_2)^{\mathcal{I}}$ the remaining R_2-edges. In the second last step we need to add all clauses (resp., clause propositions) to the tree that are satisfied by either x_1 or $\overline{x}_1$. Therefore let R^* be the transitive closure of all R_i for $1 \leq i \leq n$. For all $w \in \Delta^{\mathcal{I}_\varphi}$ s.t. $w_1 R^* w$ add w to $(x_1)^{\mathcal{I}_\varphi}$ and the same for all $w_1' R^* w$ add w to $(x_1')^{\mathcal{I}_\varphi}$. Analogously add the propositions in $\mathcal{C}_1 = \{C_j \mid x_1 \in C_j\}$ resp. $\mathcal{C}_1' = \{C_j \mid \overline{x}_1 \in C_j\}$ in the same way to the individuals $w \in (x_1)^{\mathcal{I}_\varphi}$ resp. $w \in (x_1')^{\mathcal{I}_\varphi}$ and construct the respecting clause-structures around the individuals induced by axioms (6). For each $w \in \Delta^{\mathcal{I}}$ add a transition (w, w_0) to $(S)^{\mathcal{I}_\varphi}$. Finally, for each individual $w \in \{w_0, w_1, w_1', w_s\}$ and each variable x_i we need to add w either to $(x_i)^{\mathcal{I}}$ or $(x_i')^{\mathcal{I}}$. This can be done arbitrarily and is just for satisfying the axioms (2). Also we need to built arround those states the needed nodes and edges induced by the clause axioms (6)–(9).

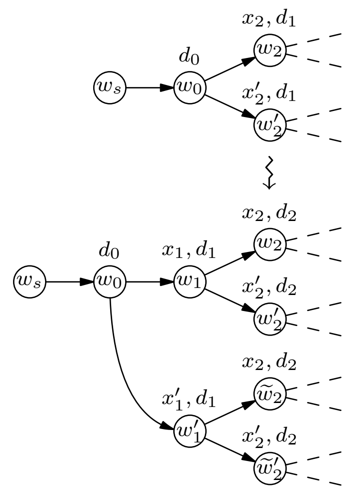

Fig. 3. Construction from $\mathcal{T}_{\chi_1}$ to $\mathcal{T}_\varphi$ in the proof of Lemma 12.

As $\chi_1 \equiv \forall x_2 \cdots \ni x_n F[x_1/\top] \equiv \top$, the "upper" subtree starting below w_1 is consistent with $x_1 \sqsubseteq \forall R_2.\exists R_3.\cdots \ni_{n+1} R_{n+1}.f'$ and now it follows from the hypothesis and construction that $\mathcal{I}_\varphi \models \mathcal{T}_\varphi$. This proof generalizes to arbitrary quantification blocks $\ni_1 x_1 \cdots \ni_n x_n$ with $\ni_i \in \{\exists, \forall\}$. $\diamond$

As the number of axioms in $\mathcal{T}$ is polynomially bounded and the terminology is consistent if and only if the quantified Boolean formula φ is satisfiable, the lemma applies. $\square$

Lemma 13. TSAT($\{\neg\}$) *is* PSPACE-*hard.*

Proof. From Lemma 2 we can simulate $\top$ and $\bot$ with fresh atomic concepts. Then the argumentation follows similarly to Lemma 12. $\square$

Lemma 14. TCSAT($\{\bot\}$) *is* coNP-*hard.*

Proof. In contrast to Lemma 12, the instances of TCSAT(I_0) consist of a concept C and a TBox $\mathcal{T} \subseteq \mathsf{Ax}(\{\bot\})$. Both do not contain the concept $\top$. Now we adapt the proof of Lemma 12 to this new setting as follows: in all axioms containing $\top$, we replace $\top$ with a fresh atomic concept t. This is unproblematic except for axiom (1), where we need to enforce d_0 to have an instance. For this purpose, we remove the axiom $\top \sqsubseteq \exists S.d_0$ from $\mathcal{T}$ and set $C = d_0$. Additionally, we need to adopt axiom (10) to $d_0 \sqsubseteq \forall R_1.\forall R_2.\cdots\forall R_n.f'$ to match the desired reduction from TAUT. Please note, that with this construction it is not possible to state a reduction from QBF-3-SAT, because an interpretation where whenever we want to branch existentially, a respective individual with neither x_i nor x_i' labeled can be added without interfering the axioms, in particular axiom (2). $\square$

5 Conclusion

With Theorem 7, we have separated the problems TSAT, TCSAT, OSAT and OCSAT for $\mathcal{ALC}$ fragments obtained by arbitrary sets of Boolean operators into tractable and intractable cases. We have shown that these problems are on the one hand for TSAT

- EXPTIME-hard whenever we allow the constant $\top$ in combination with at least conjunction or disjunction,
- EXPTIME-hard whenever all Boolean self-dual functions can be expressed,
- PSPACE-hard whenever we allow at least negation or both constants $\bot, \top$,
- trivial in all other cases.

On the other hand for the remaining three satisfiability problems we reached EXPTIME-hardness even for only disjunction or conjunction (without the constant $\top$), and got coNP-hard cases whenever we allow at least the constant $\bot$ (hence the $\bot$-reproducing cases that are trivial for TSAT drop to intractable for these problems).

According to the Figures 4 and 5, which arrange our results in Post's lattice, this classification covers all sets of Boolean operators closed under nesting.

We have also shown how our results, and the direct transfer of the results in [16] to concept satisfiability, generalise known results for the $\mathcal{FL}$ and $\mathcal{EL}$ family and other fragments of $\mathcal{ALC}$. Furthermore, due to the presence of arbitrary axioms, the overall picture differs from similar analyses for related logics in that hardness already holds for small sets of inexpressive Boolean operators.

It remains for future work to find matching upper bounds for the hardness results, to look at fragments with only existential or universal restrictions, and

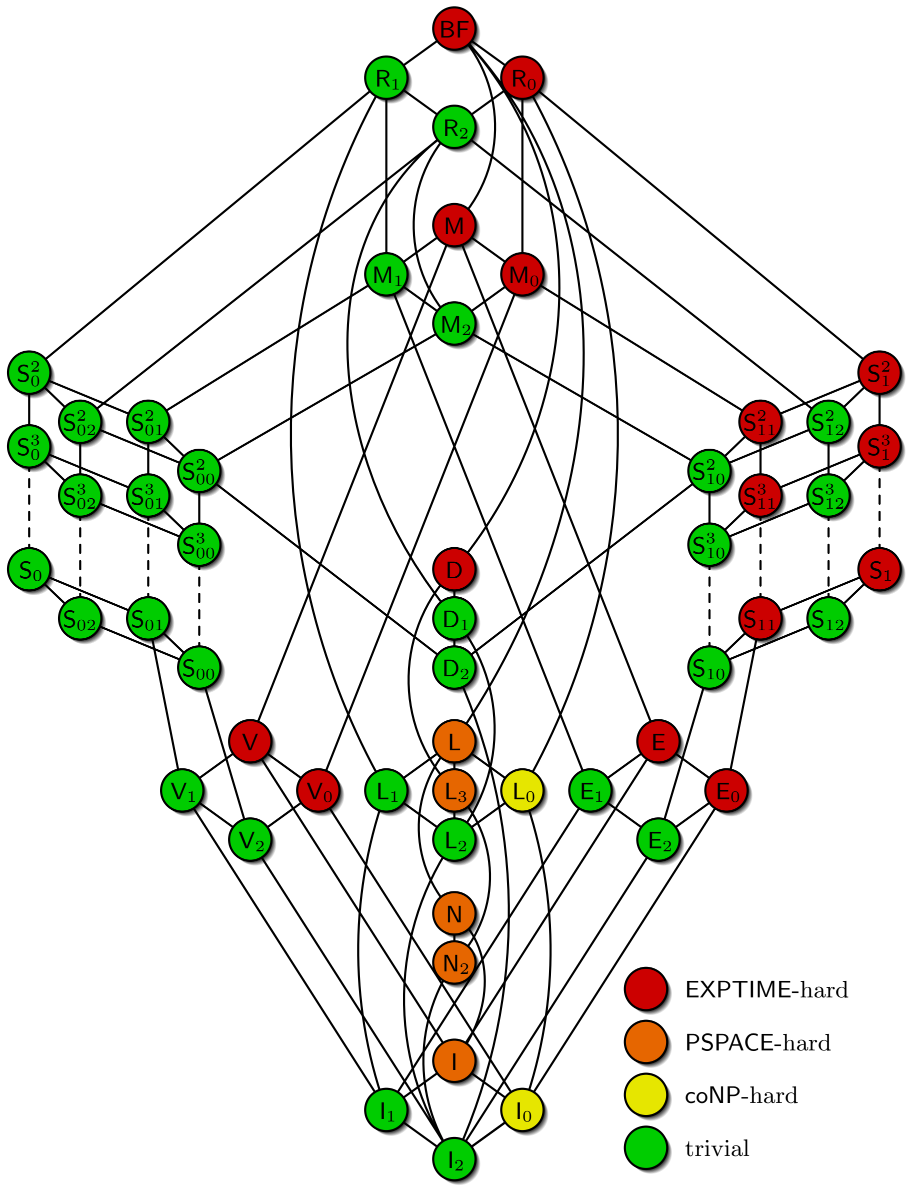

Fig. 4. Complexity for TCSAT(B), OSAT(B) and OCSAT(B).

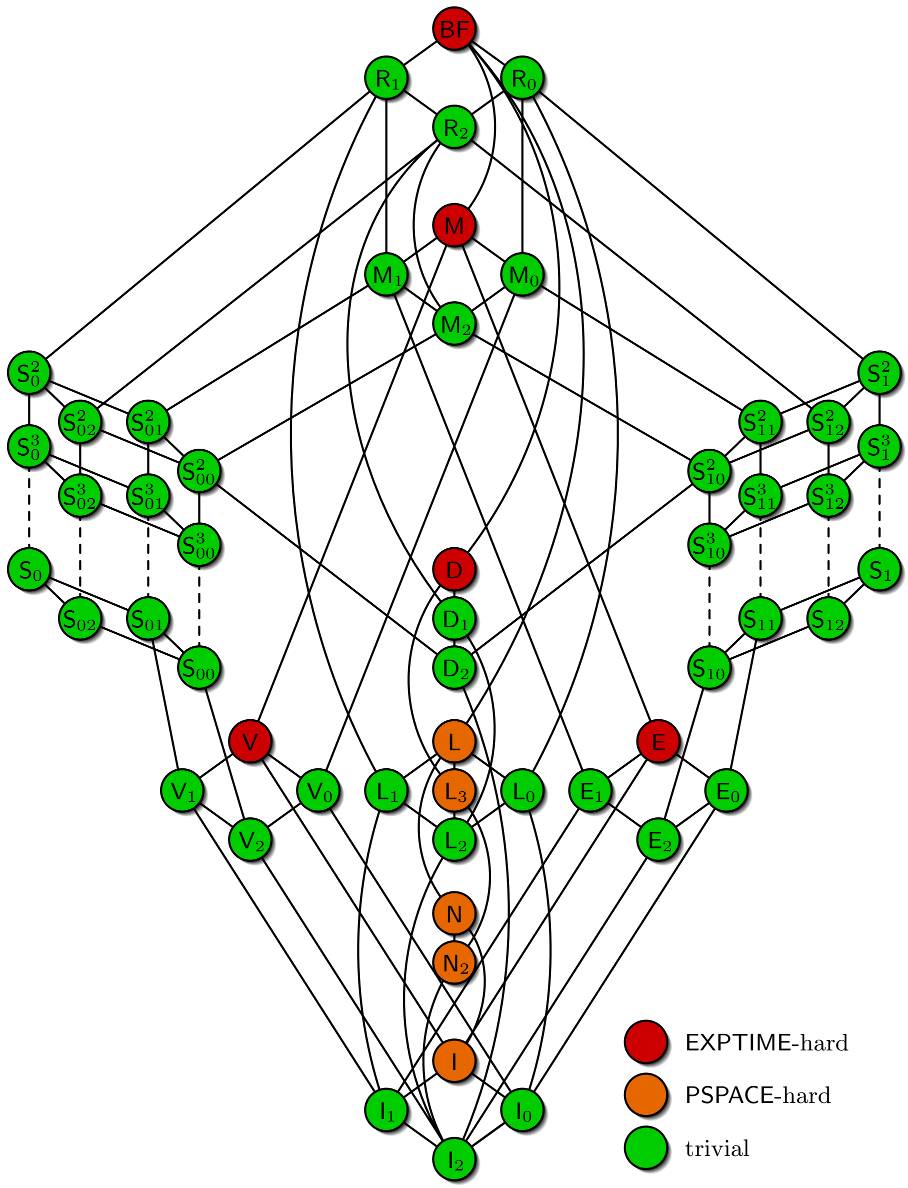

Fig. 5. Complexity for TSAT(B).

to restrict the background theories to terminologies with atomic left-hand sides of concept inclusion axioms with and without cycles. Furthermore, since the standard reasoning tasks are not always interreducible if the set of Boolean operators is restricted, a similar classification for other decision problems such as concept subsumption is pending.

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References

[1] A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. DL-Lite in the light of first-order logic. In *Proc. AAAI*, pages 361–366, 2007.

[2] A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. Adding weight to DL-Lite. In *Proc. DL*, http://CEUR-WS.org, 2009.

[3] F. Baader. Using automata theory for characterizing the semantics of terminological cycles. *Ann. Math. Artif. Intell.*, 18(2-4):175–219, 1996.

[4] F. Baader. Terminological cycles in a description logic with existential restrictions. In *Proc. IJCAI*, pages 325–330, 2003.

[5] F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope. In *Proc. IJCAI*, pages 364–369, 2005.

[6] F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope further. In *Proc. OWLED DC*, 2008.

[7] M. Bauland, T. Schneider, H. Schnoor, I. Schnoor, and H. Vollmer. The complexity of generalized satisfiability for Linear Temporal Logic. *LMCS*, 5(1), 2009.

[8] E. Böhler, N. Creignou, S. Reith, and H. Vollmer. Playing with Boolean blocks, part I: Post's lattice with applications to complexity theory. *ACM-SIGACT Newsletter*, 34(4):38–52, 2003.

[9] S. Brandt. Polynomial time reasoning in a description logic with existential restrictions, GCI axioms, and—what else? In *Proc. ECAI*, pages 298–302, 2004.

[10] D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. DL-Lite: Tractable description logics for ontologies. In *Proc. AAAI*, pages 602–607, 2005.

[11] F. M. Donini. Complexity of reasoning. In *Description Logic Handbook*, pages 96–136. Cambridge University Press, 2003.

[12] F. M. Donini, M. Lenzerini, D. Nardi, B. Hollunder, W. Nutt, and A. Marchetti-Spaccamela. The complexity of existential quantification in concept languages. *AI*, 53(2-3):309–327, 1992.

[13] F. M. Donini, M. Lenzerini, D. Nardi, and W. Nutt. The complexity of concept languages. *Inf. Comput.*, 134(1):1–58, 1997.

[14] F. M. Donini and F. Massacci. EXPTIME tableaux for $\mathcal{ALC}$. *AI*, 124(1):87–138, 2000.

[15] Robert Givan, David McAllester, Carl Wittny, and Dexter Kozen. Tarskian set constraints. *Information and Computation*, 174:105–131, 2002.

[16] Edith Hemaspaandra, Henning Schnoor, and Ilka Schnoor. Generalized modal satisfiability. *CoRR*, abs/0804.2729, 2008.

[17] M. Hofmann. Proof-theoretic approach to description-logic. In *Proc. LICS*, pages 229–237, 2005.

[18] Y. Kazakov and H. de Nivelle. Subsumption of concepts in $\mathcal{FL}_0$ for (cyclic) terminologies with respect to descriptive semantics is PSPACE-complete. In *Proc. DL*, http://www.CEUR-WS.org, 2003.

[19] H. Lewis. Satisfiability problems for propositional calculi. *Math. Sys. Theory*, 13:45–53, 1979.

[20] A. Meier, M. Mundhenk, T. Schneider, M. Thomas, V. Weber, and F. Weiss. The complexity of satisfiability for fragments of hybrid logic — Part I. In *Proc. MFCS*, volume 5734 of *LNCS*, pages 587–599, 2009.

[21] B. Nebel. Terminological reasoning is inherently intractable. *AI*, 43(2):235–249, 1990.

[22] C. H. Papadimitriou. *Computational Complexity*. Addison-Wesley, 1994.

[23] E. Post. The two-valued iterative systems of mathematical logic. *Annals of Mathematical Studies*, 5:1–122, 1941.

[24] V. R. Pratt. A practical decision method for propositional dynamic logic: Preliminary report. In *STOC*, pages 326–337. ACM, 1978.

[25] T. J. Schaefer. The complexity of satisfiability problems. In *Proc. STOC*, pages 216–226. ACM Press, 1978.

[26] Manfred Schmidt-Schauß and Gert Smolka. Attributive concept descriptions with complements. *AI*, 48(1):1–26, 1991.

[27] Henning Schnoor. *Algebraic Techniques for Satisfiability Problems*. PhD thesis, Leibniz University of Hannover, 2007.

[28] M. Y. Vardi and P. Wolper. Automata-theoretic techniques for modal logics of programs. *JCSS*, 32(2):183–221, 1986.