On Coalgebras which are Algebras

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Dedicated to Horst Herrlich on the occasion of his retirement

Abstract

The category $\mathsf{Coalg}\Sigma$ of coalgebras with respect to a (bounded) signature Σ is known to be locally finitely presentable (see [1]). We strenghten this result by showing that $\mathsf{Coalg}\Sigma$ even is a presheaf category. Moreover, we give a presentation of this category as the category of all algebras of some (many-sorted) signature (without any equations).

Σ–coalgebras, i.e., coalgebras with respect to a polynomial endofunctor H_Σ on Set, $H_\Sigma(X) = \coprod_{n<\lambda} \Sigma_n \times X^n$, with $\Sigma = (\Sigma_n)_{n<\lambda}$, a family of sets, are known to be intimately related to tree structures. On the one hand the set T_Σ of all Σ–labelled trees (see 1 below) is the underlying set of a terminal object in $\mathsf{Coalg}\Sigma$, the category of Σ–coalgebras; on the other hand, each Σ–labelled tree t is a Σ–coalgebra $\mathbb{A}_t$ in its own right (see Definition 3 below). The structural importance of the family of tree coalgebras $\mathbb{A}_t$, $t \in T_\Sigma$, already emerged in [1] where this family was shown to be a strong generator of finitely presentables in $\mathsf{Coalg}\Sigma$.

We are going to show in this note that this family even is an absolute generator, i.e., that the hom-functors determined by its members even preserve all colimits, which then leads to a representation of $\mathsf{Coalg}\Sigma$ as a presheaf–category (see also [5], where completely different methods have been used to establish such a presentation). The particular structure of the full subcategory spanned by the tree coalgebras then even allows for a simple explicit description of this presheaf category as a category of unary algebras without equations.

We start by briefly recalling some basic concepts.

1 A Σ–labelled tree is a partial function

$$t\colon \omega^* \to \Sigma$$

whose domain of definition, $\mathsf{Def}t$, has the following two properties:

(i) $\mathsf{Def}t$ contains the empty word ϵ and is prefix–closed, i.e., if $uv \in \mathsf{Def}t$ then $u \in \mathsf{Def}t$

and

(ii) if $i_1 \dots i_k \in \mathsf{Def}t$ and $t(i_1 \dots i_k)$ has arity n, then for all $j < \omega$ we have

$$i_1 \dots i_k j \in \mathsf{Def}t \text{ iff } j < n.$$

Let t be a Σ–labelled tree and $w \in \mathsf{Def}t$. By $t(w-)$ we denote "the subtree of t with root w", i.e., the tree with

$$v \in \mathsf{Def}t(w-) \Leftrightarrow wv \in \mathsf{Def}t$$

and then

$$t(w-)(v) = t(wv).$$

If $w = i \in \lambda \cap \mathsf{Def}t,\ t(i-)$ is a "maximal subtree" of t.

The set T_Σ of all Σ–labelled tree becomes a Σ–coalgebra $\mathbb{T}_\Sigma$ by means of the action α_Σ defined by

$$\alpha_\Sigma(t) = ((t(1-),\ t(2-), \dots, t(n-)),\ t(\epsilon))$$

where $t(\epsilon) \in \Sigma_n$; i.e., the action essentially assigns to a tree its family of maximal subtrees.

2 Given a Σ–coalgebra $\mathbb{C} = (C, \alpha_\mathbb{C})$ one can define, for all $c \in C$, trees $t_c \in T_\Sigma$ and elements $c_w \in C$ $(w \in \mathsf{Def}t_c)$ inductively as follows

- $c_\epsilon\colon = c$ and $t_c(\epsilon) = \sigma \Leftrightarrow \alpha_\mathbb{C}(c) = ((c_i)_{i<n}, \sigma)$
- if $v \in \mathsf{Def}t_c$ and $\alpha_\mathbb{C}(c_v) = ((c_{vi})_{i<m}, \tau)$ then $vi \in \mathsf{Def}t_c$ for all $i < m$ and $t_c(v) = \tau = (t_{c_v}(\epsilon))$.

The tree t_c defined above is called *the tree generated by c*. The resulting map $c \mapsto t_c$ then is a homomorphism $\mathbb{C} \to \mathbb{T}_\Sigma$ and, in fact, the only one (see [3]). Thus $\mathbb{T}_\Sigma$ is terminal in $\mathsf{Coalg}\Sigma$.

3 Given $t \in T_\Sigma$ the *tree coalgebra* $\mathbb{A}_t$ is defined as follows:

$$\begin{array}{lcl} \mathbb{A}_t & = & (\mathsf{Def}t, \alpha_t) \text{ with} \\ \alpha_t(w) & = & ((w1, \dots, wn), \sigma) \text{ with } \sigma = t(w) \in \Sigma_n. \end{array}$$

Recall from [1] that the collection of tree coalgebras $\mathbb{A}_t,\ t \in T_\Sigma$, is multifree on one generator; more explicitely:

For any coalgebra $\mathbb{C} = (C, \alpha_\mathbb{C})$ and any $c \in C$ there exists a homomorphism

$$h\colon \mathbb{A}_t \to \mathbb{C}$$

with $h(\epsilon) = c$, moreover here t and h are uniquely determined (t is the image of c under the unique homomorphism $(C, \alpha_\mathbb{C}) \to (T_\Sigma, \alpha_\Sigma)$, i.e., $t = t_c$).

Denote, for a coalgebra $\mathbb{C} = (C, \alpha_c)$, the map sending $c \in C$ to $h\colon A_t \to C$ with $h(\epsilon) = c$, by $\varphi_{\mathbb{C}}$. Then a straightforward calculation shows (with U the underlying functor of $\mathsf{Coalg}\Sigma$):

Corollary *The family of maps $\varphi_{\mathbb{C}}$ is a natural isomorphism*

$$\varphi\colon U \simeq \coprod_{t \in T_\Sigma} \hom(\mathbb{A}_t, -)$$

¿From this one concludes, since U is faithful and conservative:

4 Proposition *The family $(\mathbb{A}_t)_{t \in T_\Sigma}$ is strongly generating, i.e., the family of* hom*–functors*

$$\hom(\mathbb{A}_t, -),\ t \in T_\Sigma$$

is jointly faithful and jointly reflects isomorphism.

5 Proposition $\hom(\mathbb{A}_t, -)$ *preserves colimits, for each $t \in T_\Sigma$*[1].

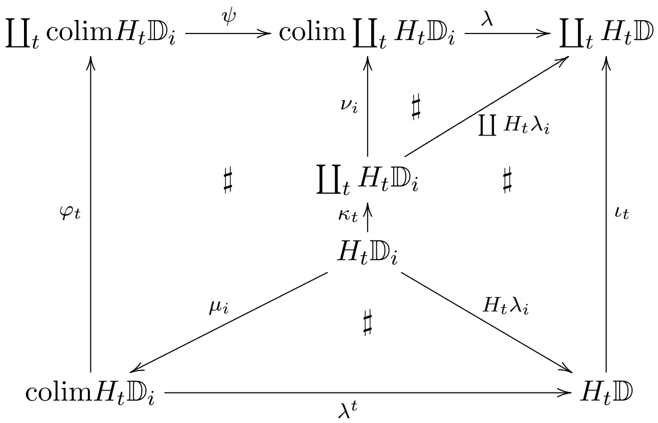
Proof: Since the functor $U\colon \mathsf{Coalg}\Sigma \to \mathsf{Set}$ is known to preserve colimits, the Corollary shows that, for each colimit cocone $(\lambda_i\colon \mathbb{D}_i \to \mathbb{D})_{i \in I}$ in $\mathsf{Coalg}\Sigma$, the canonical map $\operatorname{colim}_I \coprod_t \hom(\mathbb{A}_t, \mathbb{D}_i) \xrightarrow{\lambda} \coprod_t \hom(\mathbb{A}_t, \mathbb{D})$ is bijective. If now

$$\coprod_t \operatorname{colim}_I \hom(\mathbb{A}_t, \mathbb{D}_i) \xrightarrow{\psi} \operatorname{colim}_I \coprod_t \hom(\mathbb{A}_t, \mathbb{D}_i)$$

denotes the canonical bijection resulting from commutation of colimits, it will be enough to prove that (with the canonical maps λ^t).

$$\lambda \circ \psi = \coprod_t (\lambda^t\colon \operatorname{colim} \hom(\mathbb{A}_t, \mathbb{D}_i) \to \hom(\mathbb{A}_t, \mathbb{D}))$$

This can be read off the following commutative diagram



[1] This is a special instance of the following more general result: Let, for some functor $U\colon \mathbf{A} \to \mathbf{B}$ and some $\mathbf{B}$-object B, the family of $\mathbf{A}$-objects $(A_i)_i$ be multifree on B w.r.t. U; if U preserves colimits and B is an absolute generator, then so is the family $(A_i)_i$. (Use the equivalence $\coprod_I \hom(A_i, -) \simeq \hom(B, -) \circ U$.)

where the maps indexed by t are coproduct injection, and the maps indexed by i form the respective colimit cones (H_t is short for $\hom(\mathbb{A}_t -)$). $\square$

6 Proposition $\hom(\mathbb{A}_t, -)$ *preserves kernel pairs, for each* $t \in T_\Sigma$. *The family* $(\hom(\mathbb{A}_t, -))_{t \in T_\Sigma}$ *collectively reflects kernel pairs.*

Proof: Preservation is clear. Let now $p, q \colon \mathbb{K} \to \mathbb{C}$ be a pair of homomorphisms such that, for each $t \in T_\Sigma$,

$$\hom(\mathbb{A}_t, p),\ \hom(\mathbb{A}_t, q)\colon\ \hom(\mathbb{A}_t, \mathbb{K}) \to \hom(\mathbb{A}_t, \mathbb{C})$$

is a kernel pair of some map $f_t \colon \hom(\mathbb{A}_t, \mathbb{C}) \to X_t$. This pair is then also the kernel pair of its coequalizer which is, by Proposition 5, the map $\hom(\mathbb{A}_t, c)$ where $c \colon \mathbb{C} \to \mathbb{Q}$ is a coequalizer of (p, q). If $p', q' \colon \mathbb{L} \to \mathbb{C}$ is c's kernel pair in $\mathsf{Coalg}\Sigma$, there exists a unique $h \colon \mathbb{K} \to \mathbb{L}$ with $p'h = p$ and $q'h = q$. For each $t \in T_\Sigma \hom(\mathbb{A}_t, h) \colon \hom(\mathbb{A}_t, \mathbb{K}) \to \hom(\mathbb{A}_t, \mathbb{L})$ is a bijection (as the canonical map between two kernel pairs of $\hom(\mathbb{A}_t, c)$). Thus h is an isomorphism in $\mathsf{Coalg}\Sigma$ by Proposition 4 and (p, q) is a kernel pair. $\square$

7 Proposition $\hom(\mathbb{A}_t, -)$ *preserves regular epimorphims for each* $t \in T_\Sigma$. *The family* $(\hom(\mathbb{A}_t, -)_{t \in T_\Sigma}$ *collectively reflects regular epimorphisms.*

Proof: Preservation follows from Proposition 5. Let now $q \colon \mathbb{L} \to \mathbb{Q}$ be a homomorphism such that, for each $t \in T_\Sigma$, $\hom(\mathbb{A}_t, q)$ is surjective. Let $r, s \colon \mathbb{K} \to \mathbb{L}$ be q's kernel pair in $\mathsf{Coalg}\Sigma$ and $p \colon \mathbb{L} \to \mathbb{P}$ its coequalizer. Then $\hom(\mathbb{A}_t, q)$ is a coequalizer of $(\hom(\mathbb{A}_t, r),\ \hom(\mathbb{A}_t, s))$ as is $\hom(\mathbb{A}_t, p)$. Thus the canonical morphism $h \colon \mathbb{P} \to \mathbb{Q}$ is an isomorphism since, for each $t \in T_\Sigma$, $\hom(\mathbb{A}_t, h)$ is bijective. $\square$

Corollary *In* $\mathsf{Coalg}\Sigma$*, every epimorphism is regular.*

Proof: If e is an epimorphism, so is $\hom(\mathbb{A}_t, e)$ (since $\hom(\mathbb{A}_t, e)$ preserves colimits) for each $t \in T_\Sigma$. Hence $\hom(\mathbb{A}_t, e)$ is a regular epimorphism for each $t \in T_\Sigma$, and so is e by 7. $\square$

Corollary $\mathbb{A}_t$ *is (regularly) projective, for each* $t \in T_\Sigma$.

Using Propositions 4 to 7 we now conclude by Bunge's characterization of presheaf categories (see [2, 4])

8 Theorem *Let* $\mathbf{A}$ *be the full subcategory of* $\mathsf{Coalg}\Sigma$ *spanned by all tree coalgebras. Then*

$$\mathsf{Coalg}\Sigma \simeq \mathsf{Set}^{\mathbf{A}^{\mathsf{op}}}.$$

9 The description of $\mathsf{Coalg}\Sigma$ as a many–sorted variety of unary algebras as presented in the theorem above, can be simplified by means of the following result.

Proposition *Every homomorphism $f\colon \mathbb{A}_s \to \mathbb{A}_t$ has a unique decomposition into embeddings of maximal subtrees.*

Proof: Observe first that, given a tree t and some $w \in \mathsf{Def}t$, the map $v \mapsto wv$ is a homomorphism

$$t_w\colon \mathbb{A}_{t(w-)} \to \mathbb{A}_t$$

t_w obviously sends the root of $t(w-)$ to w. Given any homomorphism $f\colon \mathbb{A}_s \to \mathbb{A}_t$, put $w = f(\epsilon)$. Since the family $(\mathbb{A}_t)_t$ is multifree on one generator we conclude $s = t(w-)$ and $f = t_w$. Thus, the only homomorphisms between tree coalgebras are embeddings of subtrees.

If $w \in \mathsf{Def}t$ is decomposed as $w = uv$ the embedding t_w decomposes as

$$\mathbb{A}_{t(w-)} \xrightarrow{t(u-)_v} \mathbb{A}_{t(u-)} \xrightarrow{t_u} \mathbb{A}_t.$$

Thus, if $w = i_1 \dots i_k$ with $i_j \in \lambda$, we obtain a decomposition of t_w as

$$A_{t(i_1 \dots i_k -)} \xrightarrow{f_k} A_{t(i_1 \dots i_{k-1} -)} \to \cdots \to A_{t(i_1 i_2 -)} \xrightarrow{f_2} A_{t(i_1 -)} \xrightarrow{f_1} A_t$$

with $f_e = t(i_1 \dots i_{e-1}-)_{i_e}$.

This is a decomposition into embeddings of maximal subtrees; it is unique since the decomposition of words into letters is unique. □

We denote by Ω_Σ the following many-sorted signature of unary algebras:

- Sorts are all $t \in T_\Sigma$;
- Operational symbols are $\bar{t}_i\colon t \to t(i-)$ for all embeddings of maximal subtrees t_i.

Then, clearly, one has

Theorem *There is an equivalence of categories $\mathsf{Set}^{\mathbf{A}^{\mathsf{op}}} \simeq \mathsf{Alg}\Omega_\Sigma$.*

10 We can describe the resulting equivalence

$$\mathsf{Coalg}\Sigma \simeq \mathsf{Alg}\Omega_\Sigma$$

directly. First, a straightforward calculation gives the following lemma:

Lemma: *For all trees t and for all coalgebras $\mathbb{C}$ there is a bijection*

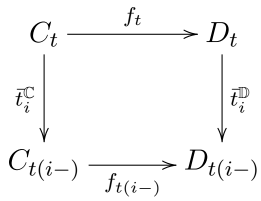
$$\hom(\mathbb{A}_t, \mathbb{C}) \simeq \{c \in C \mid t_c = t\} = !^{-1}[t]$$

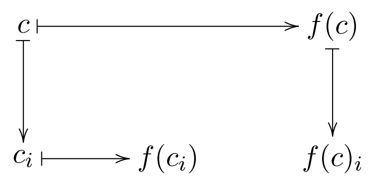
Let now $\mathbb{C} = (C, \alpha_{\mathbb{C}})$ be a Σ-coalgebra. We define an Ω_Σ-algebra $\mathbb{X}_{\mathbb{C}} = ((C_t)_t,\ (\bar{t}_i^{\mathbb{C}}))$ by

$$\begin{aligned} C_t &= \{c \in C \mid t_c = t\}, \text{ and for } c \in C_t: && (1)\\ \bar{t}_i^{\mathbb{C}}(c) &= c_i \Longleftrightarrow \alpha_{\mathbb{C}}(c) = ((c_1, \ldots, c_n),\ t(\epsilon)). && (2) \end{aligned}$$

If now, $f\colon \mathbb{C} \to \mathbb{D}$ is a homomorphism, note first that $c \in C_t$ implies $f(c) \in D_t$ (clearly f maps $!^{-1}[t] \subset C$ into $!^{-1}[t] \subset D$).

In order to prove that the resulting family of maps $f_t\colon C_t \to D_t$ is an Ω_Σ–homomorphism it remains to show that, for each t_i, the following diagram commutes.





Here c_i is determined by (see equation (2))

$$\alpha_{\mathbb{C}}(c) = ((c_1, \ldots, c_n),\ t(\epsilon))$$

while $f(c)_i$ is determined by

$$\alpha_{\mathbb{D}}(f(c)) = ((f(c)_1, \ldots, f(c)_n),\ t(\epsilon)).$$

Since f is a coalgebra homomorphism, we have

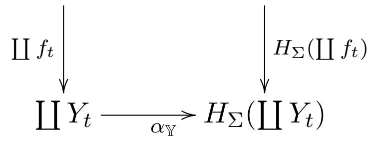
$$\alpha_{\mathbb{D}}(f(c)) = ((f(c_1),\ \ldots, f(c_n))),\ t(\epsilon))$$

thus, $f(c)_i = f(c_i)$ as required. Denote the functor $\mathsf{Coalg}\Sigma \to \mathsf{Alg}\Omega_\Sigma$ just defined by Φ.

Next we construct a functor $\Psi\colon \mathsf{Alg}\Omega_\Sigma \to \mathsf{Coalg}\Sigma$. Given $\mathbb{X} = ((X_t),\ (\bar{t}_i^{\mathbb{X}}))$, let $\mathbb{C}_{\mathbb{X}} = (C, \alpha_{\mathbb{X}})$ be the coalgebra with

$$\begin{aligned} C &= \coprod_{t \in T_\Sigma} X_t \text{ and, for } x \in X_t, && (3)\\ \alpha_{\mathbb{X}}(x) &= ((\bar{t}_1^{\mathbb{X}}(x), \ldots, \bar{t}_n^{\mathbb{X}}(x)),\ t(\epsilon)) \in H_\Sigma C. && (4) \end{aligned}$$

If $(f_t)\colon \mathbb{X} \to \mathbb{Y}$ is an Ω_Σ–homomorphism put $f = \coprod_t f_t$. We want to show that the following diagram commutes:



Choose $x \in C_t$. Then $\coprod f_t(x) = f_t(x) \in Y_t$ and

$$\begin{aligned}\alpha_{\mathbb{Y}}(\coprod f_t(x)) &= \alpha_{\mathbb{Y}}(f_t(x)) \\ &= \big((\bar{t}_1^{\mathbb{Y}}(f_t(x)), \ldots, \bar{t}_n^{\mathbb{Y}}(f_t(x))),\ t(\epsilon)\big)\end{aligned}$$

and, also (by equation (4)),

$$\begin{aligned}H_\Sigma(\coprod f_t(\alpha_{\mathbb{X}}(x))) &= H_\Sigma(\coprod f_t((\bar{t}_1^{\mathbb{X}}(x), \ldots, \bar{t}_n^{\mathbb{X}}(x)),\ t(\epsilon))) \\ &= ((f_{t(1-)}(\bar{t}_1^{\mathbb{X}}(x)), \ldots, f_{t(n-)}(\bar{t}_n^{\mathbb{X}}(x))),\ t(\epsilon)).\end{aligned}$$

Since $(f_t)_t$ is a homomorphism, we have

$$f_{t(i-)}(\bar{t}_i^{\mathbb{X}}(x)) = \bar{t}_i^{\mathbb{Y}}(f_t(x))$$

for $i = 1, \ldots, n$ as to be shown.

Now we will show, as expected, that $\Psi \circ \Phi = \mathsf{id}$ and $\Phi \circ \Psi = \mathsf{id}$. For the former take a coalgebra $\mathbb{C} = (C, \alpha_{\mathbb{C}})$ and calculate, with notations as above,

$$\Psi \circ \Phi(\mathbb{C}) = \Psi(\mathbb{X}_{\mathbb{C}}) = \mathbb{C}_{\mathbb{X}_{\mathbb{C}}} = (\coprod C_t, \alpha_{\mathbb{X}_{\mathbb{C}}})$$

Note that $\coprod C_t = C$ and, for $c \in C_t$,

$$\begin{aligned}\alpha_{\mathbb{X}_{\mathbb{C}}}(c) = ((c_1, \ldots, c_n), \sigma) &\overset{(4)}{\Longleftrightarrow} c_i = \bar{t}_i^{\mathbb{C}}(c),\ \sigma = t(\epsilon) \\ &\overset{(2)}{\Longleftrightarrow} \alpha_{\mathbb{C}}(c) = ((c_1, \ldots, c_n), \sigma).\end{aligned}$$

Thus, $\Psi \circ \Phi(\mathbb{C}) = \mathbb{C}_{\mathbb{X}_{\mathbb{C}}} = \mathbb{C}$. Obviously, $\Psi \circ \Phi(f) = f$ for homomorphisms f.

In order to prove $\Phi \circ \Psi = \mathsf{id}$, take an Ω_Σ-algebra $\mathbb{X} = ((X_t)_{t \in T_\Sigma}, (\bar{t}_i^{\mathbb{X}})_{\bar{t}_i \in \Omega_\Sigma})$. One gets

$$\Phi \circ \Psi(\mathbb{X}) = \Phi(\mathbb{C}_{\mathbb{X}}) = \mathbb{X}_{\mathbb{C}_{\mathbb{X}}} = ((C_t)_{t \in T_\Sigma}, (\bar{t}_i^{\mathbb{C}_{\mathbb{X}}})_{\bar{t}_i \in \Omega_\Sigma})$$

We start showing that, for each $t \in T_\Sigma$,

$$X_t = C_t,$$

To prove $X_t \subset C_t$ we need to show that, for each $x \in X_t$, the tree t_x generated by x in $\mathbb{C}_{\mathbb{X}}$ equals t. In fact, take $w \in \mathsf{Def}\,t \cap \mathsf{Def}\,t_x$; it first follows by induction that $x_w \in X_{t(w-)}$: Clearly $x_\epsilon = x \in X_t$. For $w = vk$ with $k < m$ and $x_v \in X_{t(v-)}$ one has, by definition of $\alpha_{\mathbb{X}}$ and t_x,

$$\alpha_{\mathbb{X}}(x_v) = ((x_{vi})_{i<m}, \tau) = ((\overline{t(v-)}_i^{\mathbb{X}}(x_v))_{i<m}, t(v)),$$

hence

$$x_w = x_{vk} = \overline{t(v-)}_k^{\mathbb{X}}(x_v) \in X_{t(v-)(k-)} = X_{t(vk-)}.$$

For these w we obtain $t_x(w) = t(w)$ since $\alpha_{\mathbb{X}}(x_w) = ((\overline{t(w-)}_i^{\mathbb{X}}(x_w))_{i<n}, t(w))$. Furthermore, both domains of definition coincide, as induction shows: Clearly $\epsilon \in \mathsf{Def}t \cap \mathsf{Def}t_x$, and for v with $\alpha_{\mathbb{X}}(x_v) = ((\overline{t(v-)}_i^{\mathbb{X}}(x_v))_{i<m}, t(v))$ we have

$$\begin{aligned} w = vk \in \mathsf{Def}t_x \ &\Leftrightarrow\ v \in \mathsf{Def}t_x,\ k < m && \text{by def. of } t_x \\ &\Leftrightarrow\ v \in \mathsf{Def}t,\ k < m && \text{by ind. hyp.} \\ &\Leftrightarrow\ vk \in \mathsf{Def}t && \text{by def. of trees.} \end{aligned}$$

Hence $t = t_x$ as required. Moreover, for $c \in C_t \subset C$ there is some $s \in T_\Sigma$ with $c \in X_s \subset C_s$, hence $t = t_c = s$. Now $X_t = C_t$ follows.

Finally, for $c \in C_t = X_t$, one has for each $\bar{t}_i$

$$\bar{t}_i^{\mathbb{C}_{\mathbb{X}}}(c) = c_i \overset{(2)}{\Longleftrightarrow} \alpha_{\mathbb{X}}(c) = ((c_1, \dots c_n), t(\epsilon)) \overset{(4)}{\Longleftrightarrow} \bar{t}_i^{\mathbb{X}}(c) = c_i.$$

Now $\Phi \circ \Psi(\mathbb{X}) = \mathbb{X}$. follows. Again, $\Phi \circ \Psi(f) = f$ for homomorphisms f is obvious[2].

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[2]Note that we didn't prove $\Phi \circ \Psi = \mathsf{id}$ literally, since we were tacitly assuming the underlying sets of an algebra to be mutually disjoint; thus $\Phi \circ \Psi$ and id are, strictly speaking, only naturally isomorphic.