Dual Adjunctions Between Algebras and Coalgebras

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Abstract

It is shown that the dual algebra functor from coalgebras to algebras has a left adjoint even if the base ring is not a field but an arbitrary commutative ring with 1. This result is proved as a corollary to a more general theorem on adjunctions between comonoids and monoids over suitable symmetric monoidal categories.

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1 Introduction

Given an R-algebra A and an R-coalgebra C (R a commutative ring with 1), one can construct the *convolution algebra* determined by the pair (C, A) (see, e.g., [4]). Fixing A one so gets a (contravariant) functor Ψ_A from coalgebras to algebras over R which, for the special case of $A = R$, is the so-called *dual algebra functor* $(-)^*\colon (\mathbf{Coalg}_R)^{\mathsf{op}} \to \mathbf{Alg}_R$ between the dual of the category $\mathbf{Coalg}_R$ of coalgebras over R, in other words, the category of comonoids in the (symmetric monoidal) category $\mathbf{Mod}_R$ of R-modules, and the category $\mathbf{Alg}_R$ of R-algebras (monoids in $\mathbf{Mod}_R$). It is an important and well known fact that, in case R is a field, the dual algebra functor has a left adjoint ([4, 10]).

The construction of the convolution algebra can be generalized in a straightforward way to any pair $(\mathcal{C}, \mathcal{M})$ consisting of a comonoid $\mathcal{C}$ and a monoid $\mathcal{M}$ over an arbitrary symmetric monoidal category $\mathbb{C}$ instead of $\mathbf{Mod}_R$ where, however, the additive structure will be missing, such that only a *convolution monoid* will arise. Fixing $\mathcal{M}$ as above we thus get functors from comonoids over $\mathbb{C}$ to (ordinary) monoids.

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These functors are shown in this note to have left adjoints under assumptions on $\mathbb{C}$ satisfied for example by $\mathbf{Mod}_R$ and **Set**, the category of sets with binary cartesian product as monoidal structure. It is somewhat amazing to see that this result is—at the same time—a wide ranging generalization of a highly non-trivial result in the module case (the existence of an adjoint of the dual algebra functor mentioned above), and of nothing more than the triviality that hom-functors have left adjoints, whenever their domains have coproducts, in the set case.

2 Monoids and comonoids

In the sequel we will assume that $\mathbb{C} = (\mathbf{C}, - \otimes -, I, a, l, r, s)$ is a symmetric monoidal category, where $a, (l, r, s)$ denote the natural isomorphisms expressing associativity (left and right unit law, symmetry) and which — except for the symmetry— we will suppress occasionally. We then have the categories $\mathbf{Mon}\mathbb{C}$ of monoids in $\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$ of comonoids in $\mathbb{C}$ defined in the obvious way (see e.g. [9]). In this note we will pay attention in particular to the following two cases:

2.1 The cartesian case. Clearly, **MonSet** is just the category **Monoids** of ordinary monoids (when **Set** is considered a monoidal category by binary products), and, somewhat more generally: if $\mathbb{C}$ is monoidal with $- \otimes - = - \times -$, the binary product, and $I = 1$, the terminal object, then $\mathbf{Mon}\mathbb{C}$ is the category of monoid objects in $\mathbf{C}$. We will call this the *cartesian case*. Forming comonoids in the cartesian case seems to be uninteresting at this stage: each $\mathbf{C}$–object C carries precisely one cartesian comonoid structure: $(C, \Delta, !)$ with $\Delta\colon C \longrightarrow C \times C$ the diagonal and $!\colon C \longrightarrow 1$ the unique morphism. Thus, in the cartesian case, $\mathbf{C} \simeq \mathbf{Comon}(\mathbf{C}, - \times -, 1)$.

2.2 The case of R-modules. For $\mathbb{C} = \mathbf{Mod}_R$, the category of modules over a commutative ring R equipped with its usual tensor product, we have that $\mathbf{MonMod}_R$ equals $\mathbf{Alg}_R$, the categories of R–algebras, while $\mathbf{ComonMod}_R =: \mathbf{Coalg}_R$ is called the *category of R–coalgebras.*

3 Convolution monoids

The following construction is a common generalization of two familiar ones, namely of the dual algebra construction as well as of the construction of powers of (ordinary) monoids. Again monoids and comonoids are understood with respect to an arbitrary symmetric monoidal category $\mathbb{C}$.

3.1 The convolution monoid functor. Given a monoid $\mathcal{M} = (M, m, e)$ and a comonoid $\mathcal{C} = (C, \mu, \epsilon)$ the homset $\mathbf{C}(C, M)$ carries the structure of an ordinary monoid—called the *convolution monoid of* $(\mathcal{C}, \mathcal{M})$—as follows:

- for $f, g\colon C \longrightarrow M$ define a product (called the *convolution product*) $f * g = C \xrightarrow{\mu} C \otimes C \xrightarrow{f \otimes g} M \otimes M \xrightarrow{m} M$.
- The unit for this multiplication is $C \xrightarrow{\epsilon} I \xrightarrow{e} M$.

1 Proposition *The construction above gives a functor*

$$\Phi\colon (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C} \longrightarrow \mathbf{Monoids}$$

such that the diagram

$$\begin{array}{ccc} (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C} & \xrightarrow{\Phi} & \mathbf{Monoids} \\ \downarrow & & \downarrow \\ \mathbf{C}^{\mathsf{op}} \times \mathbf{C} & \xrightarrow[\mathbf{C}(-,-)]{} & \mathbf{Set} \end{array}$$

commutes, where $\mathbf{Monoids} = \mathbf{Mon}(\mathbf{Set}, - \times -, 1)$ *and the unlabelled arrows are (a product of) the forgetful functors.*

3.2 Powers of monoids. In the cartesian case the convolution monoid functor can be considered a functor

$$\Phi\colon \mathbb{C}^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C} \longrightarrow \mathbf{Monoids}$$

such that the diagram

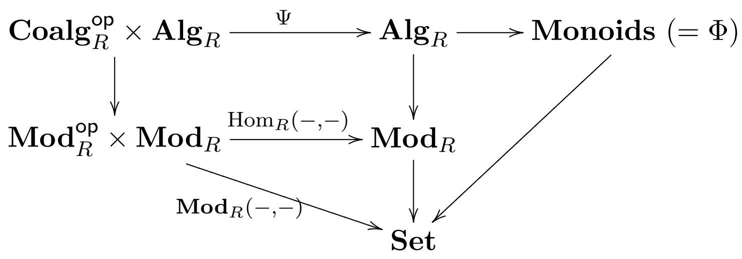
$$\begin{array}{ccc} \mathbf{C}^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C} & \xrightarrow{\Phi} & \mathbf{Monoids} \\ \downarrow & & \downarrow \\ \mathbf{C}^{\mathsf{op}} \times \mathbf{C} & \xrightarrow[\mathbf{C}(-,-)]{} & \mathbf{Set} \end{array}$$

commutes (see 2.1). Note that, in this case, convolution product and unit are given by

- $f * g = C \xrightarrow{\Delta} C \times C \xrightarrow{f \times g} M \times M \xrightarrow{m} M$,
- $C \xrightarrow{!} 1 \xrightarrow{e} M$.

In the special case of $\mathbb{C} = \mathbf{Set}$ this is nothing but the structure of M^C with pointwise multiplication, in other words, the convolution monoid of $(\mathcal{C}, \mathcal{M})$ is simply the $C's$ power of the monoid M.

3.3 Convolution algebras. In the special case of $\mathbb{C} = \mathbf{Mod}_R$ one easily shows that the construction of 3.1 makes the R–module $\mathrm{Hom}_R(C, M)$ an R–algebra; in other words: Φ can be lifted to a functor Ψ as indicated in the following diagram



We then call $\Psi(\mathcal{C}, \mathcal{M})$ the *convolution algebra* of the pair $(\mathcal{C}, \mathcal{M})$.

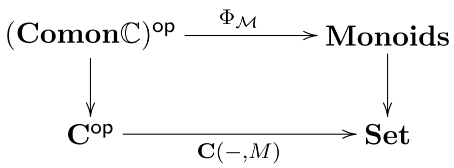
4 Some dual adjunctions

In this section we assume that the underlying category $\mathbf{C}$ of $\mathbb{C}$ is locally presentable and that the functor $C \mapsto C \otimes C$ is finitary (i.e., that it preserves directed colimits). Note that this hypothesis is satisfied in the module case (in fact, tensor power functors are finitary in any monoidally closed category, see e.g. [8]) as well as in the cartesian case provided $\mathbf{C}$ is locally finitely presentable. We then call $\mathbb{C}$ an *admissible* monoidal category.

Fixing, in the general situation of 3.1, a monoid $\mathcal{M} = (M, m, e)$ we obtain a contravariant functor from $\mathbf{Comon}\mathbb{C}$ to $\mathbf{Monoids}$, that is, a (covariant) functor

$$\Phi_{\mathcal{M}} \colon (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \longrightarrow \mathbf{Monoids}$$

such that the following diagram commutes.



2 Theorem *Let $\mathbb{C}$ be an admissible monoidal category. Then, for each monoid $\mathcal{M}$ over $\mathbb{C}$, the functor*

$$\Phi_{\mathcal{M}} \colon (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \longrightarrow \mathbf{Monoids}$$

has a left adjoint.

The main ingredience in the proof of this theorem is the following lemma from [9], which essentially is an application of elements of Makkai and Paré's Limit theorem for accessible categories (see [7]): the comonoid axioms determine three pairs of of natural transformations between accessible functors and $\mathbf{Comon}\mathbb{C}$ is the equifier of these, thus an accessible category.

3 Lemma ([9]) *For every admissible monoidal category $\mathbb{C}$ the category* $\mathbf{Comon}\mathbb{C}$ *is locally presentable.*

Proof (of the Theorem) The Special Adjoint Functor Theorem (SAFT) (see [6]) applies by the following observations: (a) $\mathbf{Comon}\mathbb{C}$ is an accessible category by the lemma above and cocomplete, thus a locally presentable category. Therefore it also is cowellpowered and has a generator (see e.g. [2]). (b) The functors $\Phi_{\mathcal{M}}$ preserve limits, since $\mathbf{Comon}\mathbb{C} \to \mathbf{C}$ preserves colimits (see [1] or [9]), hom-functors preserve limits and, finally, $\mathbf{Monoids} \to \mathbf{Set}$ creates limits. □

4.1 The case of modules. We observe first that, in this case, also the functors $\Psi_{\mathcal{M}}$ with

$$\Phi_{\mathcal{M}} = \mathbf{Coalg}_R^{\mathsf{op}} \xrightarrow{\Psi_{\mathcal{M}}} \mathbf{Alg}_R \longrightarrow \mathbf{Monoids}$$

preserve all limits. Since, for any commutative ring R, the category $\mathbf{Coalg}_R$ is locally presentable (see [8] or [9]) the argument in the proof of Theorem 2 applies and we obtain

4 Proposition *Let R be a commutative ring and $\mathcal{M} = (M, m, e)$ any R–algebra. Then the functor $\Psi_{\mathcal{M}} \colon \mathbf{Coalg}_R^{\mathsf{op}} \longrightarrow \mathbf{Alg}_R$ has a left adjoint.*

Note that, for $\mathcal{M} = R$, the functor $\Psi_{\mathcal{M}}$ is the so–called *dual algebra functor* and $\Psi_R(\mathcal{C})$ the *dual algebra* of the coalgebra $\mathcal{C}$ (see [4, 10]). Thus, Proposition 4 generalizes the well known fact that, for R a field, the dual algebra functor has a left adjoint.

4.2 The cartesian case. Theorem 2 applies to the cartesian structure on any locally finitely presentable category, that is, we have—identifying $\mathbf{K}$ and $\mathbf{Comon}(\mathbf{K}, - \times -, 1)$ as in 2.1

5 Proposition *For every monoid $\mathcal{M}$ in a locally finitely presentable category $\mathbf{K}$ the functor $\Phi_{\mathcal{M}} \colon \mathbf{K}^{\mathsf{op}} \longrightarrow \mathbf{Monoids}$ has a left adjoint.*

We here only discuss the most trivial case, that is, when $\mathbb{C}$ is $\mathbf{Set}$, because we find amazing that this proposition can be seen at the same time as a generalization of the adjointness of the dual algebra functor and the trivial fact that, for every category $\mathbf{K}$ with coproducts and every K in $\mathbf{K}$, the functor

$$\begin{array}{rccc} (-) \cdot K \colon & \mathbf{Set} & \longrightarrow & \mathbf{K} \\ & X & \longmapsto & X \cdot K \end{array}$$

is left adjoint to

$$\mathbf{K}(K, -) \colon \mathbf{K} \longrightarrow \mathbf{Set}.$$

For, if one applies this fact to $\mathbf{Monoids}^{\mathsf{op}}$, the dual to the category of (ordinary) monoids, one gets—after taking duals—that, for any monoid $\mathcal{M}$, the functor

$$\begin{array}{rccl} \mathcal{M}^{(-)}\colon & \mathbf{Set}^{\mathsf{op}} & \longrightarrow & \mathbf{Monoids} \\ & X & \longmapsto & \mathcal{M}^X \end{array}$$

which, according to 2.1 is precisely the functor $\Phi_{\mathcal{M}}$ in the present situation, is right adjoint to the (contravariant) hom-functor determined by $\mathcal{M}$.

Note that already the case $\mathbf{K} = \mathbf{Grp}$ is less trivial. Given a group G and a monoid $\mathcal{M} = (A, m, e)$ in **Grp**, that is, an Abelian group A by the Eckmann-Hilton argument, $\Phi_{\mathcal{M}}(G)$ is $hom(G, A)$ with pointwise multiplication (which is a monoid since A is Abelian). We leave it to the reader to confirm that the left adjoint of this functor assigns to a monoid N the subgroup of A^N generated by the monoid homomorphisms from N to A.

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