Hopf Monoids in semi-additive Varieties

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Abstract

We study Hopf monoids in entropic semi-additive varieties (equivalently, entropic Jónsson-Tarski varieties and categories of semimodules over a commutative semiring, respectively) with an emphasis on adjunctions related to the enveloping monoid functor and the primitive element functor. These investigations are based on the concept of the abelian core of a semi-additive variety variety and its monoidal structure in case the variety is entropic. We also complement the study of generalized finite duals.

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Introduction

The class of entropic varieties, that is, varieties whose algebraic theory is commutative, provides the canonical setting for generalizing classical Hopf algebra theory. This is shown in [20], a paper based on combining results and methods from the theories of varieties, locally presentable categories, and coalgebra, of which I had the pleasure of learning a lot in my collaboration with Jiří Adámek (see e.g. [4], [5]and [6]).

But, clearly, there are aspects of this theory, which cannot be dealt with in an arbitrary entropic variety $\mathcal{V}$. For example, the concepts of primitive element or Lie algebra require that every $\mathcal{V}$-algebra A is an internal monoid in $\mathcal{V}$, while the familiar equivalence of the various descriptions of the Sweedler dual of a k-algebra depends on the fact that the varieties of k-vector spaces are semi-additive. Since these conditions turn out to be equivalent for an entropic variety, it is natural to pay special attention to Hopf monoids in these categories.

Noting that entropic semi-additive varieties can, equivalently, be described as entropic Jónsson-Tarski varieties and as categories of semimodules over commutative semirings, respectively, (see Proposition 1.4 below) one observes that this has to some extent been done before in the recent paper [21] and [1]. Both of these papers, however, do not distinguish systematically between what can be done in any entropic variety and where the Jónsson-Tarski property really is needed. Somewhat surprisingly, moreover, some natural questions are neglected, as for example:

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1. Can one generalize the underlying Lie algebra of an algebra and the enveloping algebra of a Lie algebra?
2. Can one generalize the adjunction between bialgebras and Lie algebras, determined by the primitive element functor?

In this note it is therefore mainly dealt with these questions. In doing so we also correct some inaccuracies of these papers and posing some problems, respectively, if we could not find a sound argument for interesting claims made.

As in [20] we avoid talking about Hopf algebras, Lie algebras and bialgebras in varieties (except when these are module categories) and prefer the terms Hopf monoid, Lie monoid, and bimonoid, respectively, in order to avoid possible confusion, since the objects of the monoidal categories under consideration in this note already are called algebras.

The paper is organized as follows:

In Section 1, after briefly recalling some fundamentals about entropic varieties, we characterize entropic semi-additive as entropic Jónsson-Tarski varieties and, equivalently, as categories of semimodules over commutative semirings. Moreover, we define the abelian core of an entropic Jónsson-Tarski variety and analyze its monoidal structure. This preliminary section is complemented by a couple of results, which will be used later.

Section 2 starts with the introduction of tensor bimonoids and Lie monoids in entropic Jónsson-Tarski varieties. We then define a generalization of the familiar underlying Lie algebra of an algebra and show that the respective functor has a left adjoint, as in the case of modules.

Section 3 starts with a discussion of primitive elements in a more conceptual way as this is done in [21]; for example, the algebra of primitive elements of a bialgebra in an entropic Jónsson-Tarski variety $\mathcal{V}$ is characterized as an equalizer in the variety $\mathcal{V}$. We then construct a generalization of the familiar primitive element functor and analyze its adjunction properties.

The final Section 4 contains some results on the so-called generalized finite dual functor, which had to be left open in [20].

1 Preliminaries

1.1 Terminology

By a variety $\mathcal{V}$ we mean a finitary one-sorted variety, considered as a concrete category over **Set**, the category of sets. We will consider $\mathcal{V}$ as $\mathbf{Alg}(\Omega, \mathcal{E})$ with a set Ω of operation symbols and a set $\mathcal{E}$ of equations or, equivalently, as the category of product preserving functors $A\colon \mathcal{T} \to \mathbf{Set}$, where $\mathcal{T}$ is an algebraic theory (see e.g. [7],[11]). An n-ary term, thus, is thought of as a $\mathcal{T}$-morphism $t\colon n \to 1$ and its interpretation in an algebra A is $t^A := A(t)$. Recall that one can think of $\mathcal{T}$ as the dual of the full subcategory of $\mathcal{V}$ spanned by all finitely generated free $\mathcal{V}$-algebras Fn and, thus, of an n-ary term as a $\mathcal{V}$-homomorphism $F1 \to Fn$.

Throughout we make use of the following conventions:

1. The free $\mathcal{V}$-algebra over an n-element set will be denoted by Fn.

2. Given an element x of a $\mathcal{V}$-algebra A, the $\mathcal{V}$-homomorphism $F1 \to A$ with $1 \mapsto x$ will be denoted by x as well.

Every variety is a locally finitely presentable category. It has, in particular, (extremal episink, mono)- and (episink, extremal monosource)-factorizations in the sense of [3].

In [20] we introduced the concept of a noetherian variety. The following definition of this concept appears to be more natural.

Definition 1.1. A variety $\mathcal{V}$ will be called *noetherian*, provided that the dual of its algebraic theory $\mathcal{T}_{\mathcal{V}}$ is a noetherian category.

Recalling that an essentially small category $\mathcal{C}$ is called *noetherian*, provided that each of its objects C is noetherian in the sense, that every ascending chain of subobjects of C is stationary, this definition indeed is equivalent to the one given in [20] by the following lemma, where (2 b) is the defining property used in [20]. We assume the axiom of choice.

Lemma 1.2. *Let $\mathcal{V}$ be a variety.*

1. *For an algebra A in $\mathcal{V}$ the following are equivalent.*
   (a) *Every ascending chain of subalgebras of A is stationary.*
   (b) *Every non-empty set of subalgebras of A has a maximal element.*
   (c) *Every subalgebra of A is finitely generated.*

2. *The following conditions on $\mathcal{V}$ are equivalent.*
   (a) *Every subalgebra of a finitely generated $\mathcal{V}$-algebra A is finitely generated.*
   (b) *Every subalgebra of a finitely generated free $\mathcal{V}$-algebra Fn is finitely generated.*

Proof. The proof for the equivalences (1) is literally the same as the familiar proof for R-modules, while the proof for the equivalences (2) is literally the same as that of [10, Proposition 2.5], since this only uses conditions satisfied in any variety. □

Note that the condition that every subalgebra of the finitely generated free $\mathcal{V}$-algebra $F1$ is finitely generated (as used for $\mathcal{V} = \mathbf{Mod}_R$) is strictly weaker than (2 b) above in general as shown in [10].

Examples 1.3.

1. $\mathbf{Mod}_R$, the variety of modules over a commutative ring, is noetherian if and only if the ring R is noetherian.

2. $\mathbf{SMod}_S$, the variety of semimodules over a commutative semiring S, is noetherian if and only if the semiring S is noetherian in the sense of [14].

1.2 Entropic varieties

It is well known (see [11] or [13]), that every variety whose theory is commutative, is a symmetric monoidal closed category; following [13] we call any such symmetric monoidal closed category $\mathcal{V}$ an *entropic variety*. This tensor product of $\mathcal{V}$, called the *entropic tensor product*, is given by universal bimorphisms in the sense of [9]. In more detail, for algebras A and B in $\mathcal{V}$ their tensor product $A \otimes B$ is characterized by the fact, that there is a bimorphism $A \times B \xrightarrow{-\otimes-} A \otimes B$ over which each bimorphism $A \times B \to C$ factors uniquely as $f = g \circ (- \otimes -)$ with a homomorphism $g \colon A \otimes B \to C$. The internal hom-functor of $\mathcal{V}$ is given by the $\mathcal{V}$-algebra $[A, B]$ of all $\mathcal{V}$-homomorphisms from A to B, considered as a subalgebra of B^A. An entropic variety $\mathcal{V}$ has at most one nullary operation 0, and every $\mathcal{V}$-algebra A contains $\{0\}$ as a one-element subalgebra. In an entropic variety all operations are homomorphisms.

We will make frequent use of the following categories determined by an entropic variety $\mathcal{V}$ (see [13]):

1. $\mathsf{Mon}\mathcal{V}$, the category of $\mathcal{V}$-monoids $\mathsf{A} = (A, A \otimes A \xrightarrow{m} A, F1 \xrightarrow{e} A)$ in $\mathcal{V}$. $\mathsf{Mon}\mathcal{V}$ contains the category ${}_c\mathsf{Mon}\mathcal{V}$ of commutative monoids as a full reflective subcategory and the latter is an entropic variety.

2. $\mathsf{Comon}\mathcal{V}$, the category of $\mathcal{V}$-comonoids.

3. $\mathsf{Bimon}\mathcal{V}$, the category of $\mathcal{V}$-bimonoids.

4. $\mathsf{Hopf}\mathcal{V}$, the category of $\mathcal{V}$-Hopf monoids.

5. $\mathsf{Mod}_{\mathsf{A}}$, the category of right A-modules $(M, M \otimes A \xrightarrow{l} M)$ in $\mathcal{V}$, for any $\mathcal{V}$-monoid A; this again is an entropic variety, if the monoid A is commutative.

1.3 Entropic semi-additive varieties

A variety $\mathcal{V}$ is called *semi-additive* or *linear* if it is enriched over the monoidal closed category ${}_c$**Monoids**. Alternatively, these varieties can be characterized as being pointed (that is, they have a zero object) and having binary biproducts. These are precisely the Jónsson-Tarski varieties, whose binary Jonsson-Tarski operation + satisfies the axiom

$$(1) \qquad t(x_1, \ldots, x_n) + t(y_1, \ldots, y_n) = t(x_1 + y_1, \ldots, x_n + y_n)$$

for each n-ary operation t (see [12, 1.10.8]). In particular, + is a homomorphism.

Every such variety is a category of S-semimodules over some semiring S. In fact, thinking of an n-are operation symbol as a $\mathcal{V}$-homomorphism $F1 \xrightarrow{\omega} Fn$ one concludes that $\omega = f_1 + \cdots + f_n$ with endomorphisms $f_1, \ldots, f_n \in S$, the endomorphism monoid of the free $\mathcal{V}$-algebra $F1$, since $\mathcal{V}$ has biproducts (see [12, 1.10.8]). S also is a commutative monoid, with addition defined pointwise, by enrichment of $\mathcal{V}$ over Mon_c (that is, by the Jonsson-Tarski operation +). Since every endomorphism preserves +, S is a semiring. Now S acts on the underlying set $|A|$ of a $\mathcal{V}$-algebra by $s \cdot a := s^A(a)$ and this makes A an S-semimodule, by Equation (1). Consequently, the following holds.

Proposition 1.4. *The following are equivalent for a variety $\mathcal{V}$.*

1. *$\mathcal{V}$ is an entropic semi-additive variety.*
2. *$\mathcal{V}$ is an entropic Jónsson-Tarski variety.*
3. *$\mathcal{V}$ is the variety of S-semimodules over some commutative semiring S.*

Examples 1.5. The following varieties are entropic semi-additive varieties.

1. **Ab**, the category of abelian groups and, more generally, $\mathbf{Mod}_R$, for any commutative unital ring R.
2. ${}_c$**Monoids**, the category of all commutative monoids.
3. ${}_c\mathbf{Ring} = {}_c\mathsf{Mon}\mathbf{Ab}$, the category of all commutative unital rings.
4. ${}_c\mathbf{SRing} = {}_c\mathsf{Mon}_c\mathbf{Monoids}$, the category of all commutative unital semirings.
5. $\mathbf{SMod}_S = \mathsf{Mod}_S$ in the entropic variety ${}_c$**Monoids**; this is the category of all S-semimodules, for any commutative unital semiring S.
6. $\mathbf{SLat_0}$, the category of join-semilattices with zero.

Fact 1.6.

1. For every algebra A in an entropic Jónsson-Tarski variety $\mathcal{V}$ the triple $(A\times A \xrightarrow{+^A} A, 0)$ is a commutative internal monoid in $\mathcal{V}$ and, thus, a (commutative) monoid. This is the only internal monoid on A by the Eckmann-Hilton argument. By this construction $\mathcal{V}$ is isomorphic to the category of commutative internal monoids in $\mathcal{V}$ (see [12, 1.10.5]).
2. Every such variety $\mathcal{V}$ is isomorphic to the category $\mathsf{Mod}_{\mathsf{F1}}$ of modules of the commutative monoid $\mathsf{F1}$ (see [13]).
3. An n-fold sum $x+\cdots+x$ in an algebra A can be written as $n\cdot x$ with n the n-fold sum $1+\cdots+1 \in F1$; that is, $x+\cdots+x$ is the value of $1+\cdots+1$ under the canonical isomorphism $F1\otimes A \simeq A$.

An element x of an algebra A in a Jónsson-Tarski variety is called *invertible*, if it is invertible in the monoid $(A\times A \xrightarrow{+^A} A, 0)$, that is, if there is an element $y \in A$ with $x+y = 0$ $(= y+x)$; such an element is obviously uniquely determined and will be denoted by $-x$. $Inv(A)$ denotes the set of invertible elements of $(A, +^A, 0)$.

Lemma 1.7. *Let $\mathcal{V}$ be an entropic Jónsson-Tarski variety. Then, for every $\mathcal{V}$-algebra A, the following holds.*

1. *$Inv(A)$ is a $\mathcal{V}$-subalgebra of A with an embedding υ_A and, hence, an internal submonoid of $(A, +^A, 0)$.*

2. *Every $\mathcal{V}$-homomorphism $A \xrightarrow{f} B$ is, by restriction and corestriction, a $\mathcal{V}$-homomorphism $Inv(A) \to Inv(B)$.*

   In particular, for every $b \in B$, the homomorphism $\bar{b} := A \otimes b \colon A \simeq F1 \to A \otimes B$ restricts to a homomorphism $Inv(A) \to Inv(A \otimes B)$ and, hence, the homomorphism $\upsilon_A \otimes \upsilon_B$ factors as $Inv(A) \otimes Inv(B) \xrightarrow{\upsilon_{A,B}} Inv(A \otimes B) \hookrightarrow A \otimes B$; moreover one has $m(a \otimes b) \in Inv(A)$, for every monoid (A, m, e) in $\mathcal{V}$ and for all $a, b \in Inv(A)$.

3. *For every $\mathcal{V}$-homomorphism $A \xrightarrow{f} B$ one has $f(-x) = -f(x)$, for each $x \in Inv(A)$.*

4. *The map $Inv(A) \xrightarrow{i} Inv(A)$ with $x \mapsto -x$ is a $\mathcal{V}$-homomorphism. Consequently, $Inv(A)$ is a (commutative) internal group and, in fact the largest such contained in A.*

Proof. For every k-ary term t and invertible elements $m_1, \ldots, m_k \in A$ the element $t^A(m_1, \ldots, m_k) \in A$ is invertible, since

$$t^A(m_1, \ldots, m_k) + t^A(-m_1, \ldots, -m_k) = t^A(m_1 + (-m_1), \ldots, m_k + (-m_k)) = t^A(0, \ldots, 0) = 0.$$

Thilation shows, moreover, that the map $Inv(A) \xrightarrow{i} Inv(A)$ with $x \mapsto -x$ is a $\mathcal{V}$-homomorphism. Thus, $Inv(A)$ is an internal subgroup in $\mathcal{V}$ and, when considered as an internal monoid, it is an internal submonoid of A.

Obviously $f(-x) = -f(x)$ for every homomorphism $A \xrightarrow{f} B$ and every $x \in Inv(A)$, which proves items 2 and 4.

The rest is trivial. □

We denote by $\mathcal{V}_{\mathsf{Ab}}$ the full subcategory of $\mathcal{V}$ spanned by all $\mathcal{V}$-algebras A with $Inv(A) = A$. If $\mathcal{V} = \mathbf{Alg}(\Omega, \mathcal{E})$, then $\mathcal{V}_{\mathsf{Ab}}$ is the variety $\mathbf{Alg}(\Omega', \mathcal{E}')$ with Ω' obtained from Ω by adding a unary operation $-$, and $\mathcal{E}'$ obtained from $\mathcal{E}$ by adding the equations

1. $x + (-x) = 0$,

2. $\omega(-x_1, \ldots, -x_n) = -\omega(x_1, \ldots, x_n)$ for all n-ary operations $\omega \in \Omega$, for all $n \in \mathbb{N}$.

Obviously, $\mathcal{V}_{\mathsf{Ab}}$ is an entropic Jónsson-Tarski variety and coincides with the category of all internal groups in $\mathcal{V}$. Consequently, $\mathcal{V}_{\mathsf{Ab}}$ is an additive category (see [12, 1.10.13]) and, in fact, the largest additive subvariety of $\mathcal{V}$. Being exact as a variety, $\mathcal{V}_{\mathsf{Ab}}$ even is an abelian category (see [11, 2.6.11]). In accordance with [16] we call $\mathcal{V}_{\mathsf{Ab}}$ the *abelian core* of $\mathcal{V}$.

Proposition 1.8. *For every entropic Jónsson-Tarski variety $\mathcal{V}$ the following hold.*

1. *$\mathcal{V}_{\mathsf{Ab}}$ is a full isomorphism-closed reflective subcategory of $\mathcal{V}$.*

2. *The assignment $A \mapsto Inv(A)$ defines a functor $\mathcal{V} \xrightarrow{Inv} \mathcal{V}_{\mathsf{Ab}}$ and this is right adjoint to the embedding $\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathcal{V}$. Moreover, $\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathcal{V} \xrightarrow{Inv} \mathcal{V}_{\mathsf{Ab}} = Id$.*

3. *$\mathcal{V}_{\mathsf{Ab}}$ is closed under the entropic tensor product $-\otimes_{\mathcal{V}}-$ of $\mathcal{V}$; consequently, the entropic monoidal structure of $\mathcal{V}_{\mathsf{Ab}}$ is given by $-\otimes_{\mathcal{V}}-$ and the reflection $RF1$ of $F1$ into $\mathcal{V}_{\mathsf{Ab}}$ as the unit object.*

4. *The embedding $\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathcal{V}$ is a symmetric monoidal functor.*

Proof. $\mathcal{V}_{\mathsf{Ab}}$ is a full isomorphism-closed subcategory of $\mathcal{V}$ by the preceding lemma. Since its embedding into $\mathcal{V}$ commutes with the forgetful functors, it is an algebraic functor and, thus, has a left adjoint. Obviously every morphism of internal groups $f: G \to H$ factors over the embedding $Inv(A) \hookrightarrow A$, which shows that Inv is a coreflection. This proves items (1) and (2).

Denoting the entropic tensorproduct of $\mathcal{V}_{\mathsf{Ab}}$ by $-\otimes-$ and that of $\mathcal{V}$ by $-\otimes_{\mathcal{V}}-$, we first deduce from Lemma 1.7 that, for internal groups G and H, all elements $g \otimes_{\mathcal{V}} h \in G \otimes_{\mathcal{V}} H$ are invertible since the map $g \otimes_{\mathcal{V}} -$ is a homomorphism. Thus, $G \otimes_{\mathcal{V}} H$ is an internal group. We then have, for every triple G, H, K of internal groups in $\mathcal{V}$, $\mathcal{V}_{\mathsf{Ab}}(G \otimes H, K) \simeq \mathcal{V}_{\mathsf{Ab}}(G, [H,K]) = \mathcal{V}(G, [H,K]) \simeq \mathcal{V}(G \otimes_{\mathcal{V}} H, K) \simeq \mathcal{V}_{\mathsf{Ab}}(G \otimes_{\mathcal{V}} H, K)$, since the internal hom-functors of $\mathcal{V}$ and $\mathcal{V}_{\mathsf{Ab}}$ coincide.

To complete the proof of items (3) and (4) it remains to show that the following diagram commutes for every internal group G, where $F1 \xrightarrow{r} RF1$ is the reflection map.

(2)
$$\begin{array}{ccc} F1 \otimes_{\mathcal{V}} G & \xrightarrow{r \otimes_{\mathcal{V}} id} & RF1 \otimes_{\mathcal{V}} G \\ {\scriptstyle can_{\mathcal{V}}} \downarrow & & \| \\ G & \xleftarrow[can]{} & RF1 \otimes G \end{array}$$

But this is clear, since $F1 \otimes_{\mathcal{V}} G$ is generated by the elements $1 \otimes g$, $g \in G$, and r maps the free generator of $F1$ to the free generator of the free $\mathcal{V}_{\mathsf{Ab}}$-algebra $RF1$. □

Examples 1.9.

1. ${}_c\mathsf{Mon}\mathcal{V}_{\mathsf{Ab}}$ is isomorphic to $({}_c\mathsf{Mon}\mathcal{V})_{\mathsf{Ab}}$.
   In fact, $({}_c\mathsf{Mon}\mathcal{V})_{\mathsf{Ab}}$ is the full subcategory of ${}_c\mathsf{Mon}\mathcal{V}$, consisting of all commutative monoids (M, m, e) with $Inv(M) = M$ and, since the embedding $\mathcal{V}_{\mathsf{Ab}} \xrightarrow{E} \mathcal{V}$ is monoidal, it embeds $\mathsf{Mon}\mathcal{V}_{\mathsf{Ab}}$ into this category. Conversely, if $(M, m, e) \in {}_c\mathsf{Mon}\mathcal{V}$ satisfies $Inv(M) = M$ and $F1 \xrightarrow{r} RF1$ is the reflection of $F1$, denote by $RF1 \xrightarrow{e'} M$ the unique homomorphism with $e' \circ r = e$; then $(M, m, e') \in \mathsf{Mon}\mathcal{V}_{\mathsf{Ab}}$ (use Lemma 1.7 and Diagram (2)) and $(M, m, e) = E(M, m, e')$. In particular

   (a) $({}_c\mathbf{Monoids})_{\mathsf{Ab}} = \mathbf{Ab}$
   (b) $({}_c\mathbf{SRing})_{\mathsf{Ab}} = {}_c\mathbf{Ring}$

2. $(\mathbf{Mod}_R)_{\mathsf{Ab}} = \mathbf{Mod}_R$, for every commutative ring R and, more generally,

3. $(\mathbf{SMod}_S)_{\mathsf{Ab}} = \mathbf{Mod}_{RS}$, for every commutative semiring S, where RS is the reflection of S into the category of commutative rings[1].

[1] The embedding of the category of commutative rings into the category of commutative semirings is an algebraic functor and therefore has a left adjoint R.

This is easily seen when recalling the fact that, in every entropic variety $\mathcal{V}$, the left A-modules $(M, A \otimes M \xrightarrow{l} M)$ of a $\mathcal{V}$-monoid A are in one-to-one correspondence with $\mathcal{V}$-monoid morphisms $\mathsf{A} \xrightarrow{\phi} [M, M]$, where ϕ corresponds to l by the adjunction $- \otimes M \dashv [M, -]$ (see e.g. [20]). Thus, an S-semimodule M with $M = Inv(M)$ is a semiring homomorphism $S \xrightarrow{\phi} [M, M]$, where $[M, M]$ is the endomorphism monoid of M in ${}_c\mathbf{Monoids}$. This is a monoid in $\mathbf{Ab}$ by item (1) and, thus, ϕ corresponds to a unique ring homomorphism $RS \xrightarrow{\phi} [M, M]$, that is, to an RS-module.

4. $(\mathbf{SLat_0})_{\mathsf{Ab}} = \{0\}$.

We recall the following definitions from [20].

Definition 1.10.

1. An extremal monomorphism $S \xrightarrow{i} A$ in an entropic variety $\mathcal{V}$ is called *entropically pure* and S is called an *entropically pure subalgebra of* A, provided that $S \otimes X \xrightarrow{i \otimes X} A \otimes X$ is an extremal monomorphism, for every $\mathcal{V}$-algebra X.
2. An algebra X in an entropic variety $\mathcal{V}$ is called *entropically flat*, provided that $m \otimes X$ is an extremal monomorphism, for every extremal monomorphism m.
3. An entropic variety is called *flat*, provided that for every extremal monomorphism m in $\mathcal{V}$ its tensor square $m \otimes m$ is an extremal monomorphism.

Obviously, an entropic variety $\mathcal{V}$ is flat, if every $\mathcal{V}$-algebra X is entropically flat. The following is a partial converse in the case where $\mathcal{V}$ is a Jónsson-Tarski variety. The respective proof given in the appendix of [18] applies literally.

Lemma 1.11. *Let $\mathcal{V}$ be an entropic Jónsson-Tarski variety with surjective epimorphisms*[2]*. Then $\mathcal{V}$ is flat if and only if every $\mathcal{V}$-algebra X is entropically flat.*

For further use we note the following lemmas.

Lemma 1.12. *For every algebra A in an entropic Jónsson-Tarski variety $\mathcal{V}$ the map $A \xrightarrow{\pi} A \otimes A$ with $a \mapsto a \otimes 1 + 1 \otimes a$ is a homomorphism. and the following diagram commutes.*

(3)
$$\begin{array}{ccc} A & \xrightarrow{\pi} & A \otimes A \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi \otimes A} \\ A \otimes A & \xrightarrow[A \otimes \pi]{} & A \otimes A \otimes A \end{array}$$

Proof. π is the homomorphism $A \xrightarrow{\bar{e}} (A \otimes A) \times (A \otimes A) \xrightarrow{+} A \otimes A$, where $A \xrightarrow{\bar{e}} (A \otimes A) \times (A \otimes A)$ the unique homomorphism making the following diagram commute;

[2]In an entropic Jónsson-Tarski variety epimorphisms are not necessarily surjective (see [15, 13.43]).

here *can* denotes the canonical isomorphisms and π_1 and π_2 are the product projections.

$$
\begin{array}{ccc}
A \otimes F1 & \xrightarrow{A\otimes e} & A \otimes A \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \pi_1} \\
A & & (A \otimes A) \times (A \otimes A) \\
{\scriptstyle can}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
F1 \otimes A & \xrightarrow[e\otimes A]{} & A \otimes A
\end{array}
$$

Commutativity of Diagram (3) follows by a diagram chase. □

The following is a consequence of [20, Lemma 7].

Lemma 1.13. *Given finitely generated algebras A and B in an entropic Jónsson-Tarski variety $\mathcal{V}$, then the algebra $[A, B]$ is finitely generated, provided that $\mathcal{V}$ is noetherian.*

2 The universal envelope functor

2.1 Tensor and Lie monoids

2.1.1 Tensor bimonoids

By the standard construction of free monoids (see [18]) the free monoid TA in $\mathsf{Mon}\mathcal{V}$ over a $\mathcal{V}$-algebra A has $TA = \coprod_{n\in\mathbb{N}} A^{\otimes n}$ as its underlying $\mathcal{V}$-algebra[3] and the coproduct injection $\iota_1 : A \to TA$ as its universal morphism.

If $\mathcal{V}$ is an entropic Jónsson-Tarski variety, this $\mathcal{V}$-monoid becomes a $\mathcal{V}$-bimonoid (TA, μ, ϵ), called the *$\mathcal{V}$-tensor bimonoid*, as a corollary to Lemma 1.12 as follows.

1. $\mu : TA \to TA \otimes TA$ is the homomorphic extension of the $\mathcal{V}$-homomorphism $A \xrightarrow{\pi} A \otimes A \xrightarrow{\iota_1 \otimes \iota_1} TA \otimes TA$ to a morphism of $\mathcal{V}$-monoids.

2. $\epsilon : TA \to F1$ is the homomorphic extension the $\mathcal{V}$-homomorphism $A \xrightarrow{0} F1$ to a morphism of $\mathcal{V}$-monoids.

μ is co-associative, since Diagram (3) commutes and μ is the homomorphic extension of π. Noting that $0 \otimes x = 0$ for each $x \in X$, where X is an algebra in an entropic pointed variety, since $- \otimes x$ is a homomorphism, and that $\epsilon \circ \iota_0 = id_{F1}$ since ι_0 the is unit of TA and ϵ preserves units, one concludes that ϵ is a counit.

For every $A \in \mathcal{V}_{\mathsf{Ab}}$ the tensor bimonoid (TA, μ, ϵ) even becomes a Hopf monoid in $\mathcal{V}_{\mathsf{Ab}}$ and, thus, in $\mathcal{V}$. The required antipode acts as $a_1 \otimes \cdots a_n \mapsto (-1)^n a_n \otimes \cdots \otimes a_1$[4]. The proof is literally the same as in the case of modules.

[3] $A^{\otimes n}$ is to be understood in the obvious way: $A^{\otimes 0} = F1$ and $A^{\otimes n+1} = A \otimes A^{\otimes n}$

[4] Note that the notation $(-1)^n x$ is symbolic and short for $(-(- \cdots (-x) \cdots)$.

2.1.2 Lie monoids

Every Hopf algebra H, has an underlying Lie algebra, obtained as the underlying Lie algebra of the underlying algebra of H. This construction is not possible over an arbitrary entropic variety $\mathcal{V}$, since neither the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \tag{4}$$

cannot be expressed, nor can the underlying Lie algebra of a monoid be defined[5].

Definition 2.1. Let $\mathcal{V}$ be an entropic Jónsson-Tarski variety. A *$\mathcal{V}$-Lie monoid* is a pair $(A,[-,-])$ consisting of a $\mathcal{V}$-algebra A and a $\mathcal{V}$-bimorphism $[-,-]\colon A\times A\to A$ satisfying the identity (4) and, in addition, the identity $[x,x]=0$.
A *Lie morphism* $(A,[-,-]) \xrightarrow{f} (B,[-,-])$ is a $\mathcal{V}$-homomorphism $A \xrightarrow{f} B$ making the diagram

$$\begin{array}{ccc} A\times A & \xrightarrow{f\times f} & B\times B \\ {\scriptstyle [-,-]}\downarrow & & \downarrow{\scriptstyle [-,-]} \\ A & \xrightarrow[f]{} & B \end{array} \tag{5}$$

This defines the category $\mathsf{Lie}\mathcal{V}$ of $\mathcal{V}$-Lie monoids. $\mathsf{Lie}\mathcal{V}$. There is a forgetful functor $\mathsf{Lie}\mathcal{V} \xrightarrow{V} \mathcal{V}$.

Denoting by $[-]\colon A\otimes A\to A$ the $\mathcal{V}$-homomorphism corresponding to the bimorphism $[-,-]$ one may say, equivalently, that $(A,[-])$ is a $\mathcal{V}$-Lie monoid, if $[-]$ satisfies the axioms,

$$[x,[y\otimes z]] + [y,[z\otimes x]] + [z,[x\otimes y]] = 0 \text{ and } \quad [x\otimes x] = 0$$

In this language the Lie homomorphism axiom obviously is commutativity of Diagram (5) with $\times$ replaced by $\otimes$ and $[-,-]$ replaced by $[-]$.

Obviously, $\mathsf{Lie}\mathcal{V}$ is a Jónsson-Tarski variety and $(\mathsf{Lie}\mathcal{V})_{\mathsf{Ab}} = \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$.

2.2 The enveloping (bi)monoid of a Lie monoid

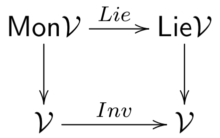
We show next that, though there is no underlying functor from $\mathcal{V}$-monoids to $\mathcal{V}$-Lie monoids for an arbitrary entropic Jónsson-Tarski variety, unless the theory of $\mathcal{V}$ admits an inverse of the Jónsson-Tarski operation $+$, that is, if $\mathcal{V} = \mathbf{Mod}_R$, and, thus, the standard categorical argument for the existence of the universal enveloping algebra UL of a Lie-algebra L does not apply[6], one can generalize the standard construction as follows, where the resulting $\mathcal{V}$-monoid even carries the structure of a $\mathcal{V}$-bimonoid, as in the case of $\mathcal{V} = \mathbf{Mod}_R$.

[5] An attempt has been made in [21] to define generalized Lie algebras in a Jónsson-Tarski variety; however, the definition given there is missing the condition that $[-,-]$ should be a bimorphism.

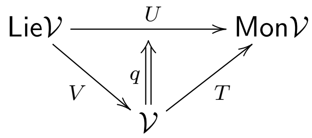
[6] An underlying functor would be algebraic and, thus, have a left adjoint.

Theorem 2.2. *There exist functors* $Lie\colon \mathsf{Mon}\mathcal{V} \to \mathsf{Lie}\mathcal{V}$ *and* $U\colon \mathsf{Lie}\mathcal{V} \to \mathsf{Mon}\mathcal{V}$*, where we call* $Lie\mathsf{A}$ *the* underlying Lie-monoid of the monoid A*, and* $U\mathsf{L}$ *the* enveloping monoid of the Lie-monoid L*, such that*

1. *the diagram commutes, that is, Lie takes its values in* $\mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$



2. *there is a natural quotient* $q\colon T \circ V \Rightarrow U$



3. *The restriction* $U_{\mathsf{Ab}} : \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathsf{Lie}\mathcal{V} \xrightarrow{U} \mathsf{Mon}\mathcal{V}$ *of* U *is left adjoint to the corestriction* $Lie_{\mathsf{Ab}}\colon \mathsf{Mon}\mathcal{V} \to \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$ *of Lie. Consequently, Lie is right adjoint to* $U_{\mathsf{Ab}} \circ R$*, where* R *is the reflection of* $\mathsf{Lie}\mathcal{V}$ *into* $\mathsf{Ab}(\mathsf{Lie}\mathcal{V}) = \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$.

4. U *factors as* $\mathsf{Lie}\mathcal{V} \xrightarrow{U_{Bi}} \mathsf{Bimon}\mathcal{V} \xrightarrow{|-|} \mathsf{Mon}\mathcal{V}$*, that is, the enveloping monoid of a Lie monoid* L *carries a bimonoid structure and this construction is functorial.*

5. U_{Ab} *factors as* $\mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \xrightarrow{U_H} \mathsf{Hopf}\mathcal{V} \to \mathsf{Mon}\mathcal{V}$*, the enveloping monoid of a Lie monoid* L *carries a Hopf monoid structure in a functorial way, provided that* $InvL = L$.

Proof. Consider, for a $\mathcal{V}$-monoid $\mathsf{A} = (A, m, e)$, the map $[-]\colon Inv(A) \otimes Inv(A) \to Inv(A)$ given by $[a \otimes b] := m(a \otimes b) - m(b \otimes a)$. This is a $\mathcal{V}$-homomorphism by item (2) of Lemma 1.7. $Lie(A, m) := (Inv(A), [-])$ then is a Lie monoid in $\mathcal{V}$ (in fact in $\mathcal{V}_{\mathsf{Ab}}$) and, for every $\mathsf{Mon}\mathcal{V}$-morphism $f\colon (A, m, e) \to (B, n, u)$ its restriction $Lie(f)$ to a homomorphism $Inv(A) \to Inv(B)$ satisfies $f([a \otimes b] = [(f \otimes f)(a \otimes b)]$, for all $a, b \in A$. This defines the functor Lie.

Given a $\mathcal{V}$-Lie monoid $(L, [-])$, form the free $\mathcal{V}$-monoid (TL, m, e) over L as in Section 2.1.1. Let $\rho\colon L \otimes L \to TL \times TL$ be the $\mathcal{V}$-homomorphism with $\pi_1 \circ \rho = m \circ \sigma \circ (\iota_1 \otimes \iota_1)$ and $\pi_2 \circ \rho = \iota_1 \circ [-]$. Then the family $\mathcal{S}_L$ of all $\mathcal{V}$-monoid morphisms $TL \xrightarrow{f_i} \mathsf{A}_i$ with $f_i \circ m \circ (\iota_1 \otimes \iota_1) = f_i \circ (+^L \circ \rho)$ has a (regular epi, monosource)-factorization $TL \xrightarrow{f_i} A_i = TL \xrightarrow{q_L} U\mathsf{L} \xrightarrow{m_i} A_i$ in $\mathsf{Mon}\mathcal{V}$, since $\mathsf{Mon}\mathcal{V}$ is a variety. If $l\colon \mathsf{L} \to \mathsf{L}'$ is a Lie-morphism, the $\mathcal{V}$-homomorphism $Tl\colon TL \to TL'$ is a monoid homomorphism and, for each $f'_i \in \mathcal{S}_{L'}$, $f'_i \circ Tl \in \mathcal{S}_L$. Thus, there exists a unique monoid morphism $Ul\colon UL \to UL'$ with $Ul \circ q_L = q_{L'} \circ TL$. This defines a functor U as well as a natural transformation $q\colon T \circ |-| \Rightarrow U$ being pointwise a quotient. We call the monoid $U\mathsf{L}$ the *enveloping monoid of* L.

Denote, for $\mathsf{L} \in \mathsf{Lie}(\mathcal{V}_{\mathsf{Ab}})$, by $\eta\colon L \to LieUL$ the $\mathcal{V}$-homomorphism $q_L \circ \iota_1$ ($L \in \mathcal{V}_{\mathsf{Ab}}$ implies $q_L \circ \iota_1(c) \in Inv(UL)$, for each $x \in L$). Since $q_L(m(\iota_1 x \otimes \iota_1 y)) = q_L(m(\iota_1 y \otimes \iota_1 x) + \iota_1[x \otimes y]) = q_L m(\iota_1 y \otimes \iota_1 x) + q_L \iota_1[x \otimes y]$ by definition of q_L, $[q_L\iota_1 x \otimes q_L \iota_1 y] = m(q_L\iota_1 x \otimes q_L\iota_1 y) - m(q_L\iota_1 x \otimes q_L\iota_1 x)$ by definition of Lie, and since q_L is a monoid morphism, one concludes that η is a $\mathsf{Lie}\mathcal{V}$-morphism.

For any Lie-morphism $\mathsf{L} \xrightarrow{f} Lie\mathsf{A}$ with $\mathsf{A} = (A, m, e) \in \mathsf{Mon}\mathcal{V}$ let $TL \xrightarrow{f^\sharp} A$ be its extension to a monoid morphism. By the definition of U this morphism factors as $f^\sharp = TL \xrightarrow{q_L} UL \xrightarrow{\tilde{f}} \mathsf{A}$ with a monoid morphism $UL \xrightarrow{\tilde{f}} \mathsf{A}$, which is the unique such morphism with $Lie\tilde{f} \circ \eta = f$. This proves the first statement of item 3. The second statement follows by composing this adjunction with the adjunction given by the embedding $(\mathsf{Lie}\mathcal{V})_{\mathsf{Ab}} \hookrightarrow \mathsf{Lie}\mathcal{V}$ and its left adjoint (see Proposition 1.8).

The monoid $U\mathsf{L}$ can be supplied with a bimonoid structure as follows. Let $u\colon L \to UL \otimes UL$ be map with $x \mapsto (q \circ \iota_1)x \otimes 1 + 1 \otimes (q \circ \iota_1)x$. In other words, with notation as in Section 2.1.1, u is the $\mathcal{V}$-homomorphism

$$L \xrightarrow{\iota_1} TL \xrightarrow{\mu} TL \otimes TL \xrightarrow{q\otimes q} UL \otimes UL = L \xrightarrow{\pi} L \otimes L \xrightarrow{\iota_1\otimes\iota_1} TL \otimes TL \xrightarrow{q\otimes q} UL \otimes UL,$$

which has $\nu := TL \xrightarrow{\mu} TL \otimes TL \xrightarrow{q\otimes q} UL \otimes UL$ as its unique extension to a $\mathcal{V}$-monoid morphism. Now the straightforward calculation

$$\begin{array}{rcl} u([x\otimes y] + m(y \otimes x)) & = & (q\otimes q)\circ(\iota_1\otimes\iota_1)\big(([x\otimes y]+m(y\otimes x))\otimes 1 + 1\otimes([x\otimes y]+m(y\otimes x))\big) \\ & = & \big(q\circ\iota_1([x\otimes y]+m(y\otimes x))\big)\otimes 1 + 1\otimes\big(q\circ\iota_1([x\otimes y]+m(y\otimes x))\big) \\ & = & \big(q\circ\iota_1(m(x\otimes y))\big)\otimes 1 + 1\otimes\big(q\circ\iota_1(m(x\otimes y))\big) \\ & = & (q\otimes q)\circ(\iota_1\otimes\iota_1)\big(m(x\otimes y)\big) \\ & = & u(m(x\otimes y)) \end{array}$$

with $x \otimes y \in L \otimes L$ proves the equation

$$u \circ m = \nu \circ \iota_1 \circ m = \nu \circ \iota_1 \circ (+ \circ \rho) = u \circ (+ \circ \rho)$$

which shows that ν belongs to $\mathcal{S}_L$. Consequently there exists a morphism of $\mathcal{V}$-monoids $UL \xrightarrow{\delta} UL \otimes UL$, such that the following diagram commutes

(6)
$$\begin{array}{ccc} TL & \xrightarrow{\quad q \quad} & UL \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\delta} \\ TL\otimes TL & \xrightarrow[q\otimes q]{} & UL\otimes UL \end{array}$$

Since the extension $\epsilon\colon TL \to F1$ of the 0-homomorphism $L \to F1$ belongs to the family $(f_i)_i$ as well, as is easily seen, ϵ factors in $\mathsf{Mon}\mathcal{V}$ as $TL \xrightarrow{q} UL \xrightarrow{\upsilon} F1$. It now follows trivially that $(TL, \mu, \epsilon) \xrightarrow{q} (UL, \delta, \upsilon)$ is a morphism of $\mathcal{V}$-bimonoids, since q is surjective. This construction is functorial: If $f\colon L \to L'$ is a Lie-morphism and Tf the corresponding monoid-morphism $TL \to TL'$, then $q_{L'} \circ Tf \in \mathcal{S}_L$, such that there is a unique monoid-morphism $Uf\colon UL \to UL'$ with $Uf \circ q_L = q_{L'} \circ Tf$. Uf is a comonoid morphism as well; it is compatible with the comonoid structures just defined, as is easily seen. This proves item 4.

Item 5 now follows literally as in the case of modules: the required antipode S is the extension of the $\mathcal{V}$-homomorphism $L \to L$ given by $x \mapsto -x$ and this is preserved by the bimonoid morphisms Uf just defined. □

3 Primitive element functors

Recall that in the classical case, where $\mathcal{V} = \mathbf{Mod}_R$, there exists a so-called the *primitive element functor* $P\colon \mathbf{Bialg}_R \to \mathbf{Lie}_R$, which is right adjoint to $U_{Bi}\colon \mathbf{Lie}_R \to \mathbf{Bialg}_R$. This cannot be generalized to arbitrary entropic varieties, since one here cannot even define primitive elements. This problem is partly addressed in [21] for Jónsson-Tarski varieties; however, due to the inaccurate definition of the generalized Lie algebras given there, the proof of one of the main results of that paper (Theorem 4.9) is incomplete. Also the problem whether the functor in this theorem gives rise to an adjunction is not considered. We here deal with this problem as follows.

3.1 Primitive elements

Definition 3.1. Let $\mathcal{V}$ be an entropic Jónsson-Tarski variety. An element p of a $\mathcal{V}$-comonoid C with comultiplication μ is called *primitive*, provided that $\mu(p) = p\otimes 1+1\otimes p$ and and $\epsilon(p) = 0$.

Note that, by definition of the comultiplication of the tensor bimonoid TA, the elements of A (more precisely, the elements $\iota_1(a)$ for $a \in A$) are primitive elements in TA.

The following conceptual description of primitive elements in bimonoids, which emerges from the following simple observation, where π is the homomorphism introduced in Lemma 1.12 and $\bar{e}$ is that one used in its proof, simplifies considerably the respective observations of [21, Section 4.3].

Proposition 3.2. *Let B be a bimonoid in an entropic Jónsson-Tarski variety $\mathcal{V}$. If E_1 denotes the equalizer of the homomorphisms $B \xrightarrow{\pi} B\otimes B$ and $B \xrightarrow{\bar{e}} (B\otimes B)\times(B\otimes B) \xrightarrow{+} B\otimes B$, and E_2 the equalizer of the homomorphisms $B \xrightarrow{\epsilon} F1$ and $B \xrightarrow{0} F1$[7], then $E_1\cap E_2$ is the set of all primitive elements. In particular, the primitive elements of B form a $\mathcal{V}$-subalgebra $Prim(\mathsf{B})$ of B.*

The assignment $\mathsf{B} \mapsto Prim(\mathsf{B})$ defines a faithful functor $Prim\colon \mathsf{Bimon}\mathcal{V} \to \mathcal{V}$.

Proof. Only the last statement requires an argument. If $f\colon \mathsf{B} \to \mathsf{B}'$ is a morphism in $\mathsf{Bimon}\mathcal{V}$ and p is a primitive element in B, then $\mu'(fp) = (f\otimes f)\circ\mu(p) = (f\otimes f)(p\otimes 1 + 1\otimes p) = fp\otimes 1 = 1\otimes fp$ by the morphism properties of f; hence f yields by restriction and corestriction a $\mathcal{V}$-homomorphism $Prim(f)\colon Prim(\mathsf{B}) \to Prim(\mathsf{B}')$. This proves functoriality of the construction $Prim$. □

Lemma 3.3. *Let (H, S) be a Hopf monoid in an entropic Jónsson-Tarski variety $\mathcal{V}$. Then*

1. *S can be restricted to a $\mathcal{V}$-homomorphisms $Inv(H) \to Inv(H)$ and $Prim(\mathsf{H}) \to Prim(\mathsf{H})$.*
2. *For each $x \in Inv(H)$ one has $Sx = -x$.*

[7] In an entropic Jónsson-Tarski variety the constant map $A \to B$ with value 0 is a homomorphism.

3. $Inv(H)$ *contains* $Prim(\mathsf{H})$ *as a* $\mathcal{V}$*-subalgebra.*

4. $Prim(\mathsf{H})$ *is an internal group in* $\mathcal{V}$*.*

The assignment $\mathsf{H} \mapsto Prim(\mathsf{H})$ *defines a faithful functor* $Prim \colon \mathsf{Hopf}\mathcal{V} \to \mathcal{V}_{\mathsf{Ab}}$*.*

Proof. S, being a homomorphism, preserves inverses by Lemma 1.7. Since the antipode S of a Hopf monoid (H, S) is a bimonoid morphism $\mathsf{H} \to \mathsf{H}^{\mathsf{op},\mathsf{cop}}$ and the antipode equation is satisfied, Sp is primitive for each primitive element p, by the simple calculation $\mu(Sp) = S \circ \sigma(\mu p) = S(1 \otimes p + p \otimes 1) = 1 \otimes Sp + Sp \otimes 1$ (recall that $+$ is commutative).

It remains to show that Sp is an additive inverse of p, for each primitive element p. In fact, $Sp + p = m(Sp \otimes 1) + m(1 \otimes p) = m(Sp \otimes 1 + 1 \otimes p) = m \circ (S \otimes id)(p \otimes 1 + 1 \otimes p) = m \circ (S \otimes id) \circ \mu(p) = e \circ \epsilon(p) = 0$. □

3.2 An adjunction between $\mathsf{Hopf}\mathcal{V}$ and $\mathsf{Lie}\mathcal{V}$

The following result generalizes the general part of the famous Milner-Moore Theorem.

Theorem 3.4. *Let* $\mathcal{V}$ *be an entropic Jónsson-Tarski variety. Then there exists a faithful functor* $\bar{P} \colon \mathsf{Hopf}\mathcal{V} \to \mathsf{Lie}\mathcal{V}$*, such that the following diagram commutes.*

$$\begin{array}{ccc} \mathsf{Hopf}\mathcal{V} & \xrightarrow{\bar{P}} & \mathsf{Lie}\mathcal{V} \\ {\scriptstyle Prim}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathcal{V}_{\mathsf{Ab}} & \hookrightarrow & \mathcal{V} \end{array}$$

$\mathsf{Hopf}\mathcal{V} \xrightarrow{P_H} \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$*, the corestriction of* $\bar{P}$*, is right adjoint to* $\mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \xrightarrow{U_H} \mathsf{Hopf}\mathcal{V}$*. Consequently,* $\bar{P}$ *is right adjoint to* $\mathsf{Lie}\mathcal{V} \xrightarrow{R} \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \xrightarrow{U_H} \mathsf{Hopf}\mathcal{V}$*, where* R *is the reflection functor.*

Proof. Given a $\mathcal{V}$-Hopf monoid H, one can define $\mathcal{V}$-homomorphisms $[-,-]_{\mathsf{H}}$ and ${}_{\mathsf{H}}[-,-]$ $Prim(\mathsf{H}) \otimes Prim(\mathsf{H}) \to Prim(\mathsf{H})$ by

$$[x, y]_{\mathsf{H}} = m(x \otimes y) + m(y \otimes Sx) \quad \text{and} \quad {}_{\mathsf{H}}[x, y] = m(x \otimes y) + m(Sy \otimes x)$$

In fact, since homomorphisms preserve invertible elements and inverses and since S and $m(x \otimes -)$ are homomorphisms, by item (2) of Lemma 1.7 the elements $m(x \otimes y)$, $m(y \otimes Sx)$ and $m(Sy \otimes x)$ are invertible, provided that x and y are primitive, hence invertible (see item (3) of Lemma 3.3). Consequently one has for primitive elements x and y

$$\begin{aligned} [x, y]_{\mathsf{H}} &= m(x \otimes y) + m(y \otimes Sx) = m(x \otimes y) + m(y \otimes (-x)) \\ &= m(x \otimes y) + (-m(y \otimes x)) = m(x \otimes y) + Sm(y \otimes x) \\ &= {}_{\mathsf{H}}[x, y] \end{aligned}$$

Now, using the equation $[x, y]_{\mathsf{H}} = m(x \otimes y) + (-m(y \otimes x))$ just shown to hold, one proves literally the same way as in the case of R-modules

1. If $x, y \in H$ are primitive, so is $[x, y]_{\mathsf{H}}$.

2. $[-,-]_{\mathsf{H}}$ satisfies Equation (4) as well as $[x, x]_{\mathsf{H}} = 0$.

This shows that $P\mathsf{H} := (Prim(\mathsf{H}), [-]_{\mathsf{H}})$ is a Lie monoid in $\mathcal{V}_{\mathsf{Ab}}$. Obviously this construction is functorial.

Recall from the construction of the tensor bimonoid that all elements of the form $\iota_1(x)$ are primitive in TL. Since $TH \xrightarrow{q} UH$ is a bimonoid morphism by commutativity of Diagram (6), it follows from Proposition 3.2 that, for each $L \in \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$, the $\mathcal{V}$-morphism $\eta = L \xrightarrow{\iota_1} TL \xrightarrow{q} LieUL = InvUL$ factors as

$$L \xrightarrow{\iota_1'} PrimTL \xrightarrow{q'} PrimUL = \bar{P}U_H(L) \hookrightarrow Inv(UL) = LieUL$$

and that $\eta' := q' \circ \iota_1'$ is a Lie-morphism.

If now, for some Hopf monoid H, $\mathsf{L} \xrightarrow{f} \bar{P}H$ is a Lie-morphism, so is $f' = L \xrightarrow{f} \bar{P}H \hookrightarrow Lie_{\mathsf{Ab}}H$, such that by item 3 of Proposition 2.2 there exists a unique monoid-morphism $\tilde{f}\colon U_{\mathsf{Ab}}L \to H$ with $f' = Lie_{\mathsf{Ab}}(\tilde{f}) \circ q \circ \iota_1$.

From $\operatorname{Im} f' \subset Prim(\mathsf{H})$ one concludes $\operatorname{Im} \tilde{f} \subset Prim(\mathsf{H})$, and this implies that the outer frame of the following diagram commutes.

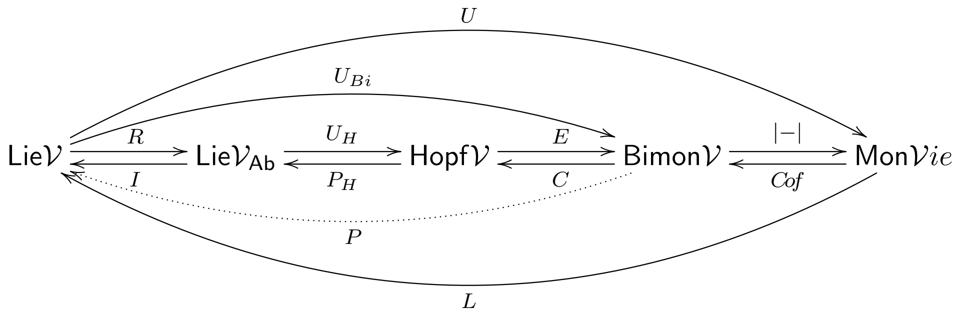
$$\begin{array}{ccccccc}
L & \xrightarrow{\iota_1} & TL & \xrightarrow{q_L} & UL & \xrightarrow{\tilde{f}} & H \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \mu_H} \\
L \otimes L & \xrightarrow[\iota_1 \otimes \iota_1]{} & TL \otimes TL & \xrightarrow[q_L \otimes q_L]{} & UL \otimes UL & \xrightarrow[\tilde{f} \otimes \tilde{f}]{} & H \otimes H
\end{array}$$

Since the left and middle cells commute by definition of μ and commutativity of Diagram (6), respectively, $\tilde{f}$ is compatible with the comultiplications. Similarly, $\tilde{f}$ preserves counits and so is a morphism in $\mathsf{Hopf}\mathcal{V}$. This shows that U_H is left adjoint to $\bar{P}$. □

Remarks 3.5. It follows from the theorem above that, as in the classical case, we also obtain a functor $\mathsf{Bimon}\mathcal{V} \to \mathsf{Lie}\mathcal{V}$ by forming the composition $I \circ P_H \circ C$, where $C\colon \mathsf{Bimon}\mathcal{V} \to \mathsf{Hopf}\mathcal{V}$ is the coreflection functor, available in every entropic variety (see [20]), and I is the embedding $\mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathsf{Lie}\mathcal{V}$. This can be considered a substitute for P, since $I \circ P_H \circ C \vdash E \circ U_H \circ R = U_{Bi} \circ I \circ R$ by composition of adjunctions and since, by construction, $U_{Bi} \circ I = E \circ U_H$. Hence, for $\mathcal{V} = \mathbf{Mod}_R$, where $I = id$, one has $I \circ P_H \circ C = P_H \circ C \vdash U_{Bi}$ and, thus, $I \circ P_H \circ C \simeq P$.

Finally, Denoting by $\mathit{Cof}\colon \mathsf{Mon}\mathcal{V} \to \mathsf{Bimon}\mathcal{V}$ the cofree bimonoid functor, that is the right adjoint of the forgetful functor $|-|\colon \mathsf{Bimon}\mathcal{V} \to \mathsf{Mon}\mathcal{V}$, available in every entropic variety as well (see [20]), one obtains $U = (|-| \circ U_{Bi}) \dashv (P \circ \mathit{Cof})$, such that $Lie \simeq P \circ \mathit{Cof}$ follows. We note that we could not find the last equation in the literature. It says in particular that, for any R-algebra A, the R-module A is isomorphic to the R-module $Prim(\mathit{Cof}\,\mathsf{A})$ of primitive elements in the cofree bialgebra over A.

For the convenience of the reader we visualize the various functors as follows.



4 Additions to the finite dual

As follows from [19] the so called semi-dualization functor $(-)^* = [-, F1] \colon \mathcal{V}^{\mathsf{op}} \to \mathcal{V}$ of an entropic variety has a left adjoint and, moreover, it is a symmetric monoidal functor and, thus, induces a *dual monoid functor* $\mathsf{Mon}\mathcal{V}^{\mathsf{op}} = (\mathsf{Comon}\mathcal{V})^{\mathsf{op}} \to \mathsf{Mon}\mathcal{V}$. This functor has a left adjoint $(-)^\bullet$, called the *generalized finite dual functor*, and it is equipped with a natural transformation $\kappa \colon (-)^\circ \Rightarrow (-)^* \circ |-|$, where $(-)^\circ$ is the functor $\mathsf{Mon}\mathcal{V} \xrightarrow{(-)^\bullet} (\mathsf{Comon}\mathcal{V})^{\mathsf{op}} \xrightarrow{||-||} \mathcal{V}^{\mathsf{op}}$.

It is desirable (see [19] for details) that the pair $((-)^\circ, \kappa)$ is a *Sweedler functor*, that is, that κ is a *monomorphic* natural transformation or, in other words, that the $\mathcal{V}$-homomorphism $\mathsf{A}^\circ \xrightarrow{\kappa_{\mathsf{A}}} A^*$ is a monomorphism, for each $\mathcal{V}$-monoid A, since in this case the functor $(-)^\bullet$ induces a functor $\mathsf{Hopf}\mathcal{V} \to (\mathsf{Hopf}\mathcal{V})^{\mathsf{op}}$.

In interesting cases, for example when $\mathcal{V} = \mathbf{Mod}_R$, the semi-dualization functor extends a rather obvious duality between certain interesting subcategories of $\mathcal{V}$ and this can be lifted to a duality on the monoid level. This, however, is not always the case as the example $\mathcal{V} = \mathbf{Set}$ shows.

We finish this paper with two contributions to these problems.

4.1 Constructing generalized finite duals

Not every generalized finite dual functor can be obtained as the lift of a Sweedler functor. One therefore is interested in conditions allowing for a construction of such "desirable" left adjoints. Various options for constructing Sweedler functors, which are equivalent in a noetherian entropic Jónsson-Tarski variety, are discussed in [20]. In the sequel $((-)^\circ, \kappa)$ always refers to this construction.

In a second step one has to make sure that $((-)^\circ, \kappa)$ is *liftable* in the sense that there exists a functor $\mathsf{Mon}\mathcal{V} \xrightarrow{(-)^\bullet} (\mathsf{Comon}\mathcal{V})^{\mathsf{op}}$ with $||-|| \circ (-)^\bullet = (-)^\circ$ such that $\mathsf{A}^\bullet$ is (in the language of [20]) an induced quotient of A by κ_{A}, for each $\mathcal{V}$-monoid A.

Finally, one has to make sure that this lifted functor $(-)^\bullet$ is left adjoint to the dual monoid functor.

In [20] it is shown that the Sweedler functor $((-)^\circ, \kappa)$ is liftable, provided that the monoidal structure on $\mathcal{V}$ has the property that the following conditions are satisfied for each $\mathcal{V}$-monoid A.

1. The morphism $\Lambda_{A,A,A} \circ (\kappa_{\mathsf{A}} \otimes \kappa_{\mathsf{A}} \otimes \kappa_{\mathsf{A}})$ is a monomorphism[8].

2. The morphism $\Lambda_{A,A} \circ (\kappa_{\mathsf{A}} \otimes \kappa_{\mathsf{A}})$ is a monomorphism.

3. The multiplication of the semidualization functor $\Lambda_{A,B}: A^* \otimes B^* \to (A \otimes B)^*$ is an isomorphism, for any two finitely generated $\mathcal{V}$-algebras A and B.

The very restrictive third condition can be avoided in a Jónsson-Tarski variety as follows. Denoting by $\Pi: F1^A \times \mathsf{A}^\circ \to F1^A \otimes \mathsf{A}^\circ$ (the restrictions of) of the canonical homomorphisms $\Pi_{A,A}$ as defined in [20, Remark 35] one first has the following lemma, whose proof is literally the same as that for [19, Lemma 45] since, by the arguments for Proposition 1.4, every n-ary operation is a sum of unary ones.

Lemma 4.1. *Let* A *be a monoid in the noetherian entropic Jónsson-Tarski variety* $\mathcal{V}$*. Then for each* $f \in \mathsf{A}^\circ$ *one has* $m^*(f) \in \Pi[F1^A \otimes \mathsf{A}^\circ] \cap \Pi[\mathsf{A}^\circ \otimes F1^A]$.

Applying now Proposition 42 and Lemma 44 of [20] one obtains

Proposition 4.2. *Assume that, for every monoid* A *in an entropic Jónsson-Tarski variety* $\mathcal{V}$*,*

1. *the homomorphisms* $\Pi_{A,A}$ *and* $\Pi_{A,A,A}$ *are injective,*

2. *the algebra* A° *is entropically pure*[9] *in* $F1^A$.

Then the Sweedler functor $((-)^\circ, \kappa)$ *is liftable.*

The lifted functor $(-)^\bullet: \mathsf{Mon}\mathcal{V} \to (\mathsf{Comon}\mathcal{V})^{\mathsf{op}}$ *is left adjoint to the dual monoid functor, provided that* $\mathcal{V}$ *is noetherian.*

It has been shown in [2] that condition 1 above is satisfied in $\mathbf{Mod}_R$, provided that R is a noetherian ring. The same has been claimed in [1] for commutative noetherian semirings; however, here is missing an argument for the fact that every finitely generated subsemimodule of a free semimodule is (in the language of that paper) *uniformly finitely generated.* Concerning condition 2 it has been shown in [2] that this is satisfied in $\mathbf{Mod}_R$, provided that R even is a hereditary noetherian ring. Nothing seems to be known for semirings.

Thus, the following problems remain open:

1. Are there noetherian commutative semirings other then rings, such that the homomorphisms $\Pi_{A,A}$ and $\Pi_{A,A,A}$ are injective?

2. For which noetherian commutative semirings is A° pure in $F1^A$, for each A?

[8] $\Lambda_{A,A,A}: LA \otimes LA \otimes LA \to L(A \otimes A \otimes A)$ denotes the canonical map.

[9] A subalgebra S of A with embedding $S \xrightarrow{i} A$ in an entropic variety $\mathcal{V}$ is *entropically pure*, provided that $B \otimes S \xrightarrow{B \otimes i} B \otimes A$ is an embedding, for every algebra B (see [20]).

4.2 The duality for finitely generated projective algebras

The following result generalizes a result, which is familiar for the case $\mathcal{V} = \mathbf{Mod}_R$, but not generalizable to arbitrary entropic varieties. ${}_{fgp}\mathcal{V}$ denotes the full subcategory of $\mathcal{V}$, spanned by all finitely generated projective algebras.

Proposition 4.3. *The semi-dualization functor of an entropic Jónsson-Tarski variety $\mathcal{V}$ can be restricted to a monoidal functor*

$$(-)^* \colon ({}_{fgp}\mathcal{V})^{\mathsf{op}} \longrightarrow {}_{fgp}\mathcal{V}$$

and this restriction provides a duality which can be lifted to a duality

$$(\mathsf{Comon}_{fgp}\mathcal{V})^{\mathsf{op}} \longrightarrow \mathsf{Mon}_{fgp}\mathcal{V}.$$

This duality is a restriction of the dual adjunction given by the dual monoid functor and the the generalized finite dual functor.

Proof. To prove the first statement, it suffices to observe that (a) $(F1)^* \simeq F1$, (b) $(-)^*$ preserves finite coproducts, since $(-)^*$ is a right adjoint and $\mathcal{V}$ has biproducts, and (c) $(-)^*$ preserves retractions by functoriality.

That this duality induces a duality on the monoid level is clear. That the inverse of the restricted dual monoid functor is the restriction of the generalized finite dual has been shown in [19]. □

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