The Essentially Equational Theory of Horn Classes

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Dedicated to Professor Dr. Dieter Pumplün on the occasion of his retirement

Abstract

It is well known that the model categories of universal Horn theories are locally presentable, hence essentially algebraic [2]. In the special case of quasivarieties a direct translation of the implicational syntax into the essentially equational one is known [1]. Here we present a similar translation for the general case, showing at the same time that many relationally presented Horn classes are in fact (equivalent to) quasivarieties.

Mathematics Subject Classification (1991). Primary 03C07, Secondary 08C15.

Key words: Universal Horn class, quasivariety, essentially equational theory.

Introduction

The class of locally finitely presentable categories is known to comprise all complete categories of finitary (many-sorted) algebras (see [3]). There is a universal syntax by which locally finitely presentable categories can be specified, namely that of an essentially equational theory, which generalizes the classical syntax of equational classes of algebras to a specification of certain classes of partial algebras (see [2]). While it is easy to specify implicationally defined classes of algebras this way (see [1]), a canonical specification for Horn classes (which are known to be locally finitely presentable categories) by an essentially equational theory seems not to be known: The univeral representation presented in [2] might be viewed unnatural: e.g., representing POS — the category of partially ordered sets and monotone maps — this way would require $\aleph_0$ sorts, whereas a 1–sorted representation is possible, as is shown in [5].

We are to provide in this note — combining ideas of [1] and [5] — an essentially equational representation of Horn classes much simpler than in [2], which moreover in many cases represents Horn classes as quasivarieties. The main idea is to introduce, for each relational symbol provided by the signature under consideration, a new sort and a family of operations corresponding to the respective projections; this enables us, in a second step, to represent (relational) implications in terms of (partial) operations and equations. For a related approach see [4].

Preliminaries

Language and notation are mainly as in [2]. To make this note as selfcontained as possible we briefly explain the basic notions.

Products of (finite) families of sets or maps are denoted by $\prod_{i=1}^{n} A_i$ or $A_1 \times \ldots \times A_n$ (respectively $\prod_{i=1}^{n} f_i$ or $f_1 \times \ldots \times f_n$) according to convenience; the product projection onto the i-th component of a (subset of a) product is denoted by π_i, with superscripts referring to the particular product if necessary. The map $B \longrightarrow \prod_{i=1}^{n} A_i$ induced by a family of maps $(g_i \colon B \longrightarrow A_i)_{i=1\ldots n}$, i.e., the map g with $\pi_i \circ g = g_i$ for each i is denoted by $\langle g_i \rangle$.

Let $\Sigma = \Sigma_{\text{op}} \dot\cup \Sigma_{\text{rel}}$ be an S*–sorted finitary signature*, i.e., S is a set of *sorts*, Σ_{op} is set of *operational symbols*, Σ_{rel} is a set of *relational symbols*, and for each symbol $\omega \in \Sigma$ there is given an *arity* $\mathsf{ar}\omega = (s_1, \ldots, s_n, s_0) \in S^{n+1}$ (for some $n \in \mathbb{N}$) . In view of the role the arities are to play we will use more suggestive notions for them as follows:

- for $\sigma \in \Sigma_{\text{op}}$ we write $s_1 \times \ldots \times s_n \to s_0$ instead of $(s_1, \ldots, s_n, s_0)$.
- for $\sigma \in \Sigma_{\text{rel}}$ we write $s_1 \times \ldots \times s_n \times s_0$ instead of $(s_1, \ldots, s_n, s_0)$.

A Σ-*structure* then is a triple $A = \Big((A_s)_{s\in S},\ (\sigma^A)_{\sigma\in\Sigma_{\text{op}}},\ (R^A)_{R\in\Sigma_{\text{rel}}}\Big)$ where $(A_s)_{s\in S}$ is a family of S-indexed sets, $\sigma^A \colon A_{s_1} \times \ldots \times A_{s_n} \to A_{s_0}$ is a map for each $\sigma \in \Sigma_{\text{op}}$ with $\mathsf{ar}\sigma = s_1 \times \ldots \times s_n \to s_0$, and $R^A \subset A_{s_1} \times \ldots \times A_{s_n} \times A_{s_0}$ is a subset for each $R \in \Sigma_{\text{rel}}$ with $\mathsf{ar}R = s_1 \times \ldots \times s_n \times s_0$. A *morphism* of Σ-structures $f \colon A \to B$ is an S-indexed family of maps $(f_s \colon A_s \to B_s)_{s\in S}$ satisfying the conditions

- $\sigma^B \circ (f_{s_1} \times \ldots \times f_{s_n}) = f_{s_0} \circ \sigma^A$ for each $\sigma \in \Sigma_{\text{op}}$ with $\mathsf{ar}\sigma = s_1 \times \ldots \times s_n \to s_0$,
- $f_{s_1} \times \ldots \times f_{s_n}[R^A] \subset R^B$ for each $R \in \Sigma_{\text{rel}}$ with $\mathsf{ar}R = s_1 \times \ldots \times s_n \times s_0$.

This defines the category $\mathsf{Str}\Sigma$ of Σ-structures which – by means of the obvious underlying functor $\mathsf{Str}\Sigma \longrightarrow \mathsf{Set}^S$ – is a concrete category over Set^S, the S-fold power of the category of sets and mappings.

The signature Σ is called *finite* provided both, S and Σ, are finite sets. As usual Σ is called a *signature of algebras* if Σ_{rel} is empty; in this case Σ-structures are simply denoted as pairs $\Big((A_s)_{s\in S},\ (\sigma^A)_{\sigma\in\Sigma_{\text{op}}}\Big)$.

The following particular types of subcategories of categories of the form $\mathsf{Str}\Sigma$ are of special interest:

Horn classes: Let there be given a set $\mathcal{I}$ of Σ–implications

$$\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n \Longrightarrow \beta$$

where $n \in \mathbb{N}$ and α_i and β are atomic Σ–formulas (i.e., either Σ_{op}-equations, or formulas $R(t_1, \ldots, t_m)$ where $R \in \Sigma_{\text{rel}}$ is an m–ary relational symbol and $t_1, \ldots, t_m$ are terms of the corresponding sorts). The full subcategory $\mathsf{Mod}(\Sigma, \mathcal{I})$ of $\mathsf{Str}\Sigma$ spanned by all Σ–structures satisfying the implications in $\mathcal{I}$ is then called a (universal) Horn class. If, for each implication in $\mathcal{I}$, the conclusion β is in fact an equation, we call the Horn class *quasivarietal*.

In case $\Sigma_{\text{rel}} = \emptyset$ the category $\mathsf{Mod}(\Sigma, \mathcal{I})$ is called a *quasivariety*. The category POS of partially ordered sets is an example of a Horn class, which is not a quasivariety. Horn classes are locally finitely presentable categories (see [2]).

Categories of partial algebras: Denote by $\mathsf{PAlg}\Sigma$ the full subcategory of $\mathsf{Str}\Sigma$ consisting of those Σ-structures A for which each relation R^A ($R \in \Sigma_{\text{rel}}$, $\mathsf{ar}R = (s_1, \ldots, s_n, s_0)$) satisfies

$$(a_1, \ldots, a_n, a_0), (a_1, \ldots, a_n, b_0) \in R^A \Longrightarrow a_0 = b_0.$$

In other words, for any A in $\mathsf{PAlg}\Sigma$ each relation R^A with arity $(s_1, \ldots, s_n, s_0)$ "is" a partial map from some subset $\mathsf{dom}R^A \subset A_{s_1} \times \ldots \times A_{s_n}$ into A_{s_0}.

Considering categories of the form $\mathsf{PAlg}\Sigma$ we might speak of *partial operational symbols* instead of relational symbols and denote their arities the same way as for (total) operational symbols; we also might write Σ_p instead of Σ_{rel} and Σ_t instead of Σ_{op} in this case, and then call Σ a *signature of partial algebras*.

Essentially equational (essentially algebraic) categories: By an *essentially equational theory* $(\Omega, \mathsf{def}, \mathcal{E})$ *with set of sorts* S we mean a S–sorted finitary signature $\Omega = \Omega_t \,\dot{\cup}\, \Omega_p$ of partial algebras together with finite sets $\mathsf{def}\omega$ of Ω_t–equations for each $\omega \in \Omega_p$ (the *domain conditions* of ω), where for $\mathsf{ar}\omega = s_1 \times \ldots \times s_n \to s_0$ the elements of $\mathsf{def}\omega$ are equations in the S-sorted variables $x_{s_1}, \ldots, x_{s_n}$, and a set $\mathcal{E}$ of Ω–equations (the *identities* of the theory). The theory is called *finite* if S and Ω are finite sets.

A partial Ω–algebra A is called a model of this theory iff

(i) the operations ω^A are total for each $\omega \in \Omega_t$;

(ii) for $\omega \in \Omega_p$ with $\mathsf{ar}\omega = s_1 \times \ldots \times s_n \to s_0$ and $(a_1, \ldots, a_n) \in A_{s_1} \times \ldots \times A_{s_n}$ one has $(a_1, \ldots, a_n) \in \mathsf{dom}\omega^A$ iff $t^A(a_1, \ldots, a_n) = t'^A(a_1, \ldots, a_n)$ for all equations $t \sim t' \in \mathsf{def}\omega$.

(iii) A satisfies the identities of $\mathcal{E}$[1].

[1]Satisfaction here is meant in the weak sense: if both sides of the equation are defined then they are equal.

These models form the full subcategory $\mathsf{Mod}(\Omega, \mathsf{def}, \mathcal{E})$ of $\mathsf{PAlg}\Omega$ (which thus is a concrete category over Set^T).

Categories of the form $\mathsf{Mod}(\Omega, \mathsf{def}, \mathcal{E})$ for essentially equational theories $(\Omega, \mathsf{def}, \mathcal{E})$ are called *essentially algebraic* (or *essentially equational*) categories. It is shown in [2] that the locally finitely presentable categories are (up to equivalence of categories) precisely the essentially equational categories.

A paradigmatic example of a (1-sorted) essentially equational category ist the category Cat of small categories and functors: the total operations are — besides the *identities* as constants — *domain* and *codomain*, the only partial (binary) operation is *composition* and def*composition* consists of the single equation *domain* (x) = *codomain* (y). Cat is not (equivalent to) a Horn class.

Each quasivariety has a canonical representation by an essentially equational theory as follows (see [1]): add to the given siganture, for each implication

$$\varrho_1 \sim \tau_1 \wedge \ldots \wedge \varrho_n \sim \tau_n \Longrightarrow \pi \sim \tau,$$

one partial operation ω with domain condition $\mathsf{def}\omega = \{\varrho_i \sim \tau_i \mid i = 1, \ldots, n\}$ together with the equations $\omega \sim \pi$ and $\omega \sim \tau$.

Results

For a given S–sorted finitary signature $\Sigma = \Sigma_{\mathrm{op}} \cup \Sigma_{\mathrm{rel}}$ we define an S^*–sorted finitary signature of algebras Ω^Σ as follows:

- $S^* = S \,\dot\cup \{s_R \mid R \in \Sigma_{\mathrm{rel}}\}$ is the set of sorts.
- $\Omega^\Sigma_{\mathrm{op}} = \Sigma_{\mathrm{op}} \,\dot\cup\, \bigcup\{\Sigma_R \mid R \in \Sigma_{\mathrm{rel}}\}$ is its set of operational symbols, where, for $R \in \Sigma_{\mathrm{rel}}$ of arity $s_1 \times \ldots \times s_n$, the set Σ_R consists of $n = n(R)$ (unary) operational symbols ${}^R\pi_i$ $(i = 1, \ldots, n)$ with arities $s_R \to s_i$; the arity of $\sigma \in \Sigma_{\mathrm{op}}$ is the one given by the signature Σ.

Note that Ω^Σ is finite provided Σ is finite.

Proposition 1 *For every S–sorted finitary signature Σ there exist a finitary S^*–sorted signature of algebras Ω^Σ, a set $\mathcal{I}^\Sigma$ of Ω^Σ–implications, and functors*

$$\mathsf{Str}\Sigma \xrightarrow{G^\Sigma} \mathsf{Mod}(\Omega^\Sigma, \mathcal{I}^\Sigma) \xrightarrow{F^\Sigma} \mathsf{Str}\ \Sigma$$

such that $F^\Sigma \circ G^\Sigma = 1$ and $G^\Sigma \circ F^\Sigma \simeq 1$.

If the signature Σ is finite, so is the signature Ω^Σ and also the set of implications $\mathcal{I}^\Sigma$.

The above proposition — being only a slight generalization of [2, 3.20] — is included here, as well as its proof, because it serves as a basis for the (proofs of) the following results.

Proposition 2 *Let Σ be an S–sorted finitary signature and $\mathsf{Mod}(\Sigma, \mathcal{J}) \subset \mathsf{Str}\Sigma$ a quasivarietal Horn class. Then there exists a set $\mathcal{I}^{\mathcal{J}}$ of Ω^{Σ}–implications (of the same cardinality as $\mathcal{J}$) such that G^{Σ} and F^{Σ} can be restricted to an equivalence between $\mathsf{Mod}(\Sigma, \mathcal{J})$ and the S^*–sorted quasivariety $\mathsf{Mod}(\Omega^{\Sigma}, \mathcal{I}^{\Sigma} \cup \mathcal{I}^{\mathcal{J}})$.*

Proposition 3 *Let Σ be an S–sorted finitary signature and $\mathsf{H} = \mathsf{Mod}(\Sigma, \mathcal{J}) \subset \mathsf{Str}\Sigma$ an arbitrary universal Horn class. Then there exists an essentially equational theory $(\Omega^{\mathsf{H}}, \mathsf{def}, \mathcal{E}^{\mathsf{H}})$ with set of sorts S^* such that H and $\mathsf{Mod}(\Omega^{\mathsf{H}}, \mathsf{def}, \mathcal{E}^{\mathsf{H}})$ are equivalent.*

If Σ is a finite signature and the set $\mathcal{J}$ of implications is finite, then $(\Omega^{\mathsf{H}}, \mathsf{def}, \mathcal{E}^{\mathsf{H}})$ is a finite theory.

Proofs

Proof of Proposition 1. For $\Sigma = \Sigma_{\text{op}} \cup \Sigma_{\text{rel}}$ with set of sorts S we have $S^* = S \,\dot\cup\, \Sigma_{\text{rel}}$ and $\Omega^{\Sigma} = \Sigma_{\text{op}} \,\dot\cup\, \bigcup\{\Sigma_R \mid R \in \Sigma_{\text{rel}}\}$. Define $\mathcal{I}^{\Sigma}$ as

$$\mathcal{I}^{\Sigma} = \left\{ {}^R\pi_1 x \sim {}^R\pi_1 y \;\wedge \ldots \wedge\; {}^R\pi_n x \sim {}^R\pi_n y \Longrightarrow x \sim y \mid R \in \Sigma_{\text{rel}} \right\}$$

Given a Σ–structure

$$A = \Big((A_s)_{s \in S},\ (\sigma^A)_{\sigma \in \Sigma_{\text{op}}},\ (R^A)_{R \in \Sigma_{\text{rel}}} \Big)$$

define

$$G^{\Sigma} A = \Big((\bar{A}_t)_{t \in S^*},\ (\sigma^{\bar{A}})_{\sigma \in \Sigma_{\text{op}}},\ ({}^R\pi_i^{\bar{A}})_{{}^R\pi_i \in \Sigma_R,\ R \in \Sigma_{\text{rel}}} \Big)$$

by

- $\bar{A}_t \colon = \begin{cases} A_s & \text{if} \quad t = s \in S \\ R^A & \text{if} \quad t = s_R,\ R \in \Sigma_{\text{rel}} \end{cases}$
- $\sigma^{\bar{A}} \colon = \sigma^A$
- ${}^R\pi_i^{\bar{A}} \colon = \prod_{i=1}^{n(R)} A_{s_i} \supset R^A \xrightarrow{\pi_i} A_{s_i}$ i.e., ${}^R\pi_i^{\bar{A}}$ is the (restricted) product projection.

$G^{\Sigma} A$ then obviously satisfies all implications in $\mathcal{I}^{\Sigma}$. G^{Σ} becomes a functor if one defines, for a $\mathsf{Str}\Sigma$–morphism $h = (h_s)_{s \in S} \colon A \longrightarrow B$, $G^{\Sigma} h = \big(\bar{h}_t \colon \bar{A}_t \longrightarrow \bar{B}_t\big)_{t \in S^*}$ by

$$\bar{h}_t = \begin{cases} h_s \ : A_s \longrightarrow B_s & \text{if} \quad t = s \in S \\ \prod_{i=1}^{n(R)} h_{s_i} \ : R^A \longrightarrow R^B & \text{if} \quad t = s_R,\ R \in \Sigma_{\text{rel}}{}^2 \end{cases}$$

[2]here, clearly, $\bar{h}_{s_R}$ is meant to be $\prod h_{s_i}$ restricted and corestricted to R^A and R^B respectively.

Now define, for any $(\Omega^\Sigma, \mathcal{I}^\Sigma)$–model

$$A = \big((A_t)_{t\in S^*},\ (\sigma^A)_{\sigma\in\Omega^\Sigma_{\mathrm{op}}}\big),$$

the Σ–model

$$F^\Sigma A = \big((\tilde{A}_s)_{s\in S},\ (\sigma^{\tilde{A}})_{\sigma\in\Sigma_{\mathrm{op}}},\ (R^{\tilde{A}})_{R\in\Sigma_{\mathrm{rel}}}\big)$$

by

- $\tilde{A}_s \colon = A_s$,
- $\sigma^{\tilde{A}} \colon = \sigma_A$,
- $R^{\tilde{A}} \colon =$ im $\langle {}^R\pi_i^A\rangle$ i.e., $R^{\tilde{A}}$ is the image of the map $\langle {}^R\pi_i^A\rangle \colon A_{s_R} \longrightarrow \prod_{i=1}^{n(R)} A_{s_i}$.

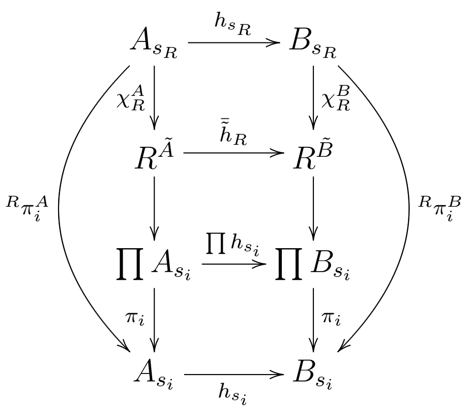
F^Σ becomes a functor if we define

$$F^\Sigma\big((A_t \xrightarrow{h_t} B_t)_{t\in S^*}\big) = (A_s \xrightarrow{h_s} B_s)_{s\in S}.$$

It is obvious that the equation $F^\Sigma \circ G^\Sigma = 1$ holds. Also for $A = \big((A_t)_{t\in S^*},\ (\sigma^A)_{\sigma\in\Omega^\Sigma}\big)$ and $G^\Sigma \circ F^\Sigma A = \big((\bar{\tilde{A}}_t)_{t\in S^*}\big),\ \big((\sigma^{\bar{\tilde{A}}})_{\sigma\in\Omega^\Sigma}\big)$ one obviously has

- $\bar{\tilde{A}}_s = A_s$ for each $s \in S$
- $\sigma^{\bar{\tilde{A}}} = \sigma^A$ for each $\sigma \in \Sigma_{\mathrm{op}}$

For $R \in \Sigma_{\mathrm{rel}}$ we have $\bar{\tilde{A}}_{s_R} = R^{\tilde{A}} = \mathrm{im}\big\langle\big({}^R\pi_i^A\big)_i\big\rangle$. Since A satisfies the implications of $\mathcal{I}^\Sigma$, the map $\big\langle\big({}^R\pi_i^A\big)_i\big\rangle$ is monic, hence a bijection $\chi_R^A \colon A_{s_R} \xrightarrow{\sim} R^{\tilde{A}}$. Similarly, for $h = (h_t)_{t\in S^*} \colon A \longrightarrow B$ in $\mathsf{Mod}(\Omega^\Sigma, \mathcal{I}^\Sigma)$ and $G^\Sigma \circ F^\Sigma h = (\bar{\tilde{h}})_{t\in S^*}$ one has $\bar{\tilde{h}}_s = h_s$ for each $s \in S$. For $R \in \Sigma_{\mathrm{rel}}$ the map $\bar{\tilde{h}}_{S_R} \colon R^{\tilde{A}} \longrightarrow R^{\tilde{B}}$ is the (co/restriction) of $\prod_{i=1}^{n(R)} h_{s_i}$. Hence, in the following diagram all cells commute.



Thus, the S^*–sorted maps $\chi_t^A \colon A_t \longrightarrow \tilde{\bar{A}}_t$ with

$$\chi_t^A = \begin{cases} 1_{A_s} & \text{if} \quad t = s \in S \\ \chi_R^A & \text{if} \quad t = s_R,\ R \in \Sigma_{\mathrm{rel}} \end{cases}$$

are isomorphisms $A \longrightarrow G^\Sigma F^\Sigma A$, natural in A. ◇

Lemma *Let Σ be a S–sorted finitary signature and $\alpha = Q(t_1, \ldots t_n)$ an atomic Σ–formula with Q a relational symbol of arity $s_1 \times \ldots \times s_n$. Let $\mathcal{E}_\alpha$ be the set of Ω^Σ–equations*

$$\mathcal{E}_\alpha = \left\{ {}^{Q}\pi_1 \sim t_1, \ldots, {}^{Q}\pi_n \sim t_n \right\}.$$

Then the following holds:

(i) If $\bar{a} = (a_1, \ldots, a_m)$ is an assignment to variables in the Σ–structure A such that $\big(t_1^A(\bar{a}), \ldots, t_n^A(\bar{a})\big) \in Q^A$, then there exists an assignment $\bar{a}^* = (b, a_1, \ldots, a_m)$ in the $(\Omega^\Sigma, \mathcal{I}^\Sigma)$–model $G^\Sigma A$ fulfilling all equations in $\mathcal{E}_\alpha$.

(ii) Let $\bar{b} = (b, a_1, \ldots, a_m)$ be an assignment to variables in the $(\Omega^\Sigma, \mathcal{I}^\Sigma)$–model A such that $\bar{b}$ fulfills all equations in $\mathcal{E}_\alpha$; then $\big(t_1^{F^\Sigma A}(\bar{b}^\bullet), \ldots, t_n^{F^\Sigma A}(\bar{b}^\bullet)\big) \in Q^{F^\Sigma A}$ for the assignment $\bar{b}^\bullet = (a_1, \ldots, a_m)$ in the in the Σ–structure $F^\Sigma A$.

Proof $x_1, \ldots, x_m$ denote the S–sorted variables needed by $t_1, \ldots, t_n$, x_i of sort s_i.

Let $A = \big((A_s),\ (\sigma^A)_\sigma,\ (R^A)_R\big)$ be a Σ–structure and $\bar{a} = (a_1, \ldots, a_m)$ with $a_i \in A_{s_i}$ an assignment to the variables $x_1, \ldots, x_m$, such that $\big(t_i^A(\bar{a}), \ldots, t_n^A(\bar{a})\big) \in Q^A$. Then $\bar{a}^* = (b,\ a_1, \ldots, a_n)$ with $b = \big(t_1(\bar{a}), \ldots, t_n(\bar{a})\big)$ is an assignment to the S^*–sorted variables $(x,\ x_1, \ldots, x_m)$ with x of sort s_Q in the Ω^Σ–model $G^\Sigma A$. To prove that $\bar{a}^*$ fulfills all equations in $\mathcal{E}_\alpha$ now is trivial.

Conversely, if $\bar{b} = (b, a_1, \ldots, a_m)$ is an assignment to the S^*–sorted variables $(x, x_1, \ldots, x_m)$ with x of sort s_Q in the $\mathsf{Mod}(\Omega^\Sigma, \mathcal{I}^\Sigma)$–model $A = \big((A_t)_t,\ (\sigma^A)_\sigma\big)$ fulfilling all equations in $\mathcal{E}_\alpha$ a straightforward calculation shows that for the assignment $\bar{b}^\bullet = (a_1, \ldots, a_m)$ to the S–sorted variables $x_1, \ldots, x_m$ in the Σ–structure $F^\Sigma A$ one has $\big(t_1^{F^\Sigma A}(\bar{b}^\bullet),\ \ldots, t_n^{F^\Sigma A}(\bar{b}^\bullet)\big) \in Q^{F^\Sigma A}$. ◇

Proof of Proposition 2. For each implication

$$J\colon\ \alpha_1 \wedge \ldots \wedge \alpha_k \Longrightarrow s \sim t$$

we consider the sets $\mathcal{E}_{\alpha_i}$ of Ω^Σ–equations as in the Lemma where applicable. If $\alpha_i = (s_i \sim t_i)$ is an equation we define $\mathcal{E}_{\alpha_i}$ to be $\{s_i \sim t_i\}$. By e_i we denote the conjunction of the equations of $\mathcal{E}_{\alpha_i}$, i.e.,

$$e_i = \begin{cases} \left({}^{R_i}\pi_1 \sim t_1^i \right) \wedge \ldots \wedge \left({}^{R_i}\pi_{n_i} \sim t_{n_i}^i \right) & \text{if} \quad \alpha_i = R_i(t_1^i, \ldots t_{n_i}^i) \\ s_i \sim t_i & \text{if} \quad \alpha_i = (s_i \sim t_i) \end{cases}$$

Let now $\bar{J}$ be the implication

$$\bar{J}\colon e_1 \wedge e_2 \wedge \ldots \wedge e_k \Longrightarrow s \sim t$$

and $\mathcal{I}^{\mathcal{J}}$ the set of all implications $\bar{J}$ for $J \in \mathcal{J}$. It remains to show that G^Σ maps $\mathsf{H} = \mathsf{Mod}(\Sigma, \mathcal{J})$ into $\mathsf{Mod}(\Omega^\Sigma, \mathcal{I}^\Sigma \cup \mathcal{I}^{\mathcal{J}})$ and that F^Σ maps this category back into H.

Let $A = \big((A_s)_s,\ (\sigma^A)_\sigma,\ (R^A)_R\big)$ be in H. It suffices to show that $G^\Sigma A$ satisfies all implications of the form $\bar{J}$. Let now $\bar{b}$ be an assignment to the variables needed for $\bar{J}$ in $G^\Sigma A$ satisfying the premis $\bar{J}$. Then $\bar{b}$ satisfies all equations in $\mathcal{E}_{\alpha_i}$ for each $i = 1, \ldots, k$. By (ii) of the Lemma there results an assignment $\bar{b}^\bullet$ in $F^\Sigma G^\Sigma A = A$ such that each atomic formula α_i is satisfied by $\bar{b}^\bullet$. Since J holds in A we conclude $s^A(\bar{b}^\bullet) = t^A(\bar{b}^\bullet)$, hence ${}^{G^\Sigma A}s(\bar{b}) = {}^{G^\Sigma A}t(\bar{b})$, since $\bar{b}$ and $\bar{b}^\bullet$ differ only in (not) assigning values to variables of the sorts s_{R_i} which don't occur in s and t respectively.

Conversely, let $A = \big((A_t)_t,\ (\sigma^A)_\sigma\big)$ be in $\mathsf{Mod}(\Omega^\Sigma, \mathcal{I}^\Sigma \cup \mathcal{I}^{\mathcal{J}})$ and $\bar{a}$ an assignment to the variables needed in $J \in \mathcal{J}$ in the Σ–structure $F^\Sigma A$, such that each atomic formula α_i occuring in the premis of J is satisfied by $\bar{a}$. By (i) of the Lemma there exists an assignment $\bar{a}^*$ in $G^\Sigma F^\Sigma A$ fulfilling the equations in $\mathcal{E}_{\alpha_i}$ for all i. Since $G^\Sigma F^\Sigma A$ and A are isomorphic by Proposition 1 and A satisfies the implications in $\mathcal{I}^{\mathcal{J}}$ so does $G^\Sigma F^\Sigma A$; hence we conclude

$$s^{G^\Sigma F^\Sigma A}(\bar{a}^*) = t^{G^\Sigma F^\Sigma A}(\bar{a}^*),\ \text{thus}\ s^{F^\Sigma A}(\bar{a}) = t^{F^\Sigma A}(\bar{a}). \qquad \diamond$$

Before considering the general case of an arbitrary Horn class $\mathsf{H} \subset \mathsf{Str}\Sigma$ we recall from the introduction the description of the quasivariety $\mathsf{Mod}(\Omega^\Sigma, \mathcal{I}^\Sigma)$ by means of an essentially equational theory: $(\tilde{\Omega}^\Sigma, \mathsf{def}, \mathcal{E}^\Sigma)$ is the essentially equational theory with

- S^* as set of sorts,
- $\tilde{\Omega}^\Sigma_t = \Omega^\Sigma_{\text{op}}$
- $\tilde{\Omega}^\Sigma_p = \{\tilde{R} \mid R \in \Sigma_{\text{rel}}\}$, where for each $R \in \Sigma_{\text{rel}}$ of arity $s_1 \times \ldots \times s_n$ the corresponding $\tilde{R}$ is considered as a (binary) operational symbol of arity $s_R \times s_R \to s_R$ with domain condition

$$\mathsf{def}\tilde{R} = \left\{\ {}^R\pi_i x \sim\ {}^R\pi_i y \mid i = 1, \ldots, n\right\}.$$

- $\mathcal{E}^\Sigma = \bigcup\limits_{R \in \Sigma_{\text{rel}}} \mathcal{E}_R$ with $\mathcal{E}_R = \left\{\tilde{R}(x,y) \sim x, \tilde{R}(x,y) \sim y\right\}$

It is well known (and easy to see) that the obvious forgetful functor

$$\mathsf{Mod}(\tilde{\Omega}^\Sigma, \mathsf{def}, \mathcal{E}^\Sigma) \longrightarrow \mathsf{Mod}(\Omega^\Sigma)$$

yields an isomorphism

$$\mathsf{Mod}(\tilde{\Omega}^\Sigma, \mathsf{def}, \mathcal{E}^\Sigma) \longrightarrow \mathsf{Mod}(\Omega^\Sigma, \mathcal{I}^\Sigma).$$

Proof of Proposition 3. We continue as in the preceding remark and enlarge the theory $(\tilde{\Omega}^{\Sigma}, \mathsf{def}, \mathcal{E}^{\Sigma})$ to $(\Omega^{\mathsf{H}}, \mathsf{def}, \mathcal{E}^{\mathsf{H}})$ as follows:

- $\Omega^{\mathsf{H}}_t :\, = \tilde{\Omega}^{\Sigma}_t = \Omega^{\Sigma}_{\mathrm{op}}$
- $\Omega^{\mathsf{H}}_p :\, = \tilde{\Omega}^{\Sigma}_p \,\dot{\cup}\, \{\tilde{J} \mid J \in \mathcal{J}\} = \Sigma_{\mathrm{rel}} \dot{\cup}\, \{\tilde{J} \mid J \in \mathcal{J}\}$ where for each implication

  $$J\colon \alpha_1 \wedge \ldots \wedge \alpha_k \Longrightarrow \beta$$

  in $\mathcal{J}$ (with $\alpha_i = R_i(t^i_1, \ldots, t^i_{n_i})$, $\beta = Q(t_1, \ldots, t_m)$ and needed variables $x_1, \ldots, x_l$ of sorts $s_1, \ldots, s_l \in S$) $\tilde{J}$ is a partial operational symbol with arity

  $$s_{R_1} \times \ldots \times s_{R_k} \times s_1 \times \ldots \times s_l \to s_Q{}^3$$

  with domain condition $\mathsf{def}\tilde{J} = \bigcup_{i=1}^{k} \mathcal{E}_{\alpha_i}$,
- $\mathcal{E}^{\mathsf{H}} = \mathcal{E}^{\Sigma} \cup \mathcal{E}^{\mathcal{J}}$ with $\mathcal{E}^{\mathcal{J}} :\, = \bigcup_{J \in \mathcal{J}} \Big\{ {}^{Q}\pi_i \tilde{J} \sim t_i \mid i = 1, \ldots, m \Big\}$.

Observe first that the obvious forgetful functor

$$V\colon \mathsf{Mod}(\Omega^{\mathsf{H}}, \mathsf{def}, \mathcal{E}^{\mathsf{H}}) \longrightarrow \mathsf{Mod}(\tilde{\Omega}^{\Sigma}, \mathsf{def}, \mathcal{E}^{\Sigma})$$

is full: if $h = (h_t)_t\colon VA \longrightarrow VB$ is a homomorphism in $\mathsf{Mod}(\tilde{\Omega}^{\Sigma}, \mathcal{E}^{\Sigma})$ and $J \in \mathcal{J}$ it is to be shown that, for each $\bar{a} = (b_1, \ldots b_k,\ a_1, \ldots, a_l)$ fulfilling all equations in $\mathsf{def}\tilde{J}$ in the algebra A, $h(\bar{a})$ fulfills these equations in B and also the homomorphism condition

$$h_Q \circ \tilde{J}^A(\bar{a}) = \tilde{J}^B \circ h_{s_{R_1}} \times \ldots \times h_{s_{R_k}} \times h_{s_1} \times \ldots \times h_{s_l}(\bar{a}) = \tilde{J}^B h(\bar{a}).$$

To prove this, consider $\bar{a} = (b_1, \ldots, b_k,\ a_1, \ldots, a_l) \in \mathsf{dom}\ \tilde{J}^A$ with b_i of sort s_{R_i}, a_j of sort s_j. $\bar{a} \in \mathsf{dom}\ \tilde{J}^A$ implies $b_i = {}^{R_i}\pi^A_{j_i}(\bar{a}) = (t^i_{j_i})^A(\bar{a})$ for all i, j_i; since h is a morphism in $\mathsf{Mod}(\Omega^{\Sigma}, \mathcal{I}^{\Sigma})$ one has

$$h_{s_{R_i}}(t^i_{j_i})^A(\bar{a}) = (t^i_{j_i})^B\big(h(\bar{a})\big) \text{ for all } i, j_i,$$

and therefore

$${}^{R_i}\pi^B_{j_i}\big(h(\bar{a})\big) = {}^{R_i}\pi^B_{j_i}\big((h_{s_{R_i}}(b_i)\big) = (t^i_{j_i})^B\big(h(\bar{a})\big),$$

thus, $h(\bar{a}) \in \mathsf{dom}\tilde{J}^B$. Now, for $\bar{a} \in \mathsf{dom}\tilde{J}^A$, one concludes for all terms t_i of sorts s'_i occuring in the conclusion β:

$$\begin{aligned} {}^{Q}\pi^B_i \tilde{J}^B\big(h(\bar{a})\big) &= t^B_i\big(h(\bar{a})\big) \\ &= h_{s'_i}\big(t^A_i(\bar{a})\big) \\ &= h_{s'_i}\big(\,{}^{Q}\pi^A_i \tilde{J}^A(\bar{a})\big) \\ &= {}^{Q}\pi^B_i h_Q\big(\tilde{J}^A(\bar{a})\big) \end{aligned}$$

[3]omit here s_{R_i} if α_i is of the form $t_1 \sim t_2$

Since VB satisfies the implications in $\mathcal{I}^\Sigma$, h is a homomorphism.

Next, we lift G^Σ and F^Σ along V to obtain functors G and F such that the diagram

$$\begin{array}{ccccc}
\mathsf{Mod}(\Sigma,\mathcal{J}) & \xrightarrow{G} & \mathsf{Mod}(\Omega^H,\mathsf{def},\mathcal{E}^H) & \xrightarrow{F} & \mathsf{Mod}(\Sigma,\mathcal{J}) \\
\downarrow & & \downarrow{\scriptstyle V} & & \downarrow \\
\mathrm{Str}\Sigma & \xrightarrow[G^\Sigma]{} & \mathsf{Mod}(\Omega^\Sigma,\mathcal{I}^\Sigma) & \xrightarrow[F^\Sigma]{} & \mathsf{Str}\Sigma
\end{array}$$

commutes, which immediately yields $F \circ G = 1$ and, since V is full and faithful, also $G \circ F \simeq 1$.

Let $A = \big((A_s)_s,\ (\sigma^A)_\sigma,\ (R^A)_R\big)$ satisfy all implications $J \in \mathcal{J}$. Consider

$$G^\Sigma A = \big((\bar{A}_t),\ (\bar{\sigma}^A)_{\sigma\in\Omega_t^\Sigma},\ (\tilde{R}^A)_{R\in\Sigma_{\mathrm{rel}}}\big)$$

(we identify, for the sake of simplicity, $\mathsf{Mod}(\tilde{\Omega}^\Sigma, \mathsf{def}, \mathcal{E}^\Sigma)$ and $\mathsf{Mod}(\Omega^\Sigma, \mathcal{I}^\Sigma)$).
Define $GA\colon = \big((\bar{A}_t),\ (\bar{\sigma}^A)_{\sigma\in\Omega_t^\Sigma},\ (\tilde{R}^A)_{R\in\Sigma_{\mathrm{rel}}},\ (\tilde{J}^A)_{J\in\mathcal{J}}\big)$, where for

$$J\colon R_1(t_1^1,\dots,t_{n_1}^1) \wedge \dots \wedge R_k(t_1^k,\dots,t_{n_k}^k) \Longrightarrow Q(t_1,\dots,t_m) \qquad (*)$$

in $\mathcal{J}$ with S–sorted variables $x_1,\dots x_l$ of sorts $s_1,\dots s_l$ needed for the $t_{j_i}^j, t_i$, the partial operation

$$R_1^A \times \dots \times R_k^A \times \prod_{i=1}^{l} A_{s_i} \supset \mathsf{dom}\tilde{J}^A \xrightarrow{\tilde{J}^A} Q^A$$

with

$$\mathsf{dom}\tilde{J}^A = \{(b_1,\dots,b_k,\bar{a}) \mid b_j \in R_j^A, \bar{a} \in \prod_{i=1}^{l} A_{s_i},\ b_j = \big(t_1^j(\bar{a}),\dots,t_{n_j}^j(\bar{a})\big)\}$$

is defined by

$$\tilde{J}^A(b_1,\dots,b_k,\bar{a}) = \big(t_1(\bar{a}),\dots,t_m(\bar{a})\big)$$

Then any $(b_1,\dots,b_k,\bar{a}) \in \prod_{j=1}^{k} R_j^A \times \prod_{i=1}^{l} A_{s_i}$ (with $b_j = (b_1^j,\dots,b_{n_j}^j)$) satisfies all equations in $\mathsf{def}\tilde{J}$ iff, for all j, i_j, the equation $b_{i_j}^j = {}^{R_j}\pi_{i_j}^A(b_j) = (t_{ij}^j)^A(\bar{a})$ holds, i.e., iff $(b_1,\dots,b_k\bar{a}) \in \mathsf{dom}\tilde{J}^A$, and, if so, $\tilde{J}^A(b_1,\dots,b_k,\bar{a}) \in Q^A$ since A satisfies the implication J.

GA satisfies the identities of $\mathcal{E}^\mathcal{J}$, hence of $\mathcal{E}^\mathsf{H}$, since for each $(b_1,\dots,b_k,\bar{a})$ as above one has

$$\begin{aligned}
{}^Q\pi_i^A \tilde{J}^A(b_1,\dots,b_k,\bar{a}) &= {}^Q\pi_i\big(t_1^A(\bar{a}),\dots,t_m^A(\bar{a})\big) \\
&= t_i^A(\bar{a}) = t_i^A(b_1,\dots,b_k,\bar{a}).
\end{aligned}$$

Since V is full we can define $Gh = G^\Sigma h$ for morphisms h in $\mathsf{Mod}(\Sigma, \mathcal{J})$.

In order to define F we only need to show that, for each A in $\mathsf{Mod}(\Omega^{\mathsf{H}}, \mathsf{def}, \mathcal{E}^{\mathsf{H}})$, $A = \big((A_t)_t,\ (\sigma^A)_\sigma, (\tilde{R}^A)_R, (\tilde{J}^A)_J\big)$, the Σ-structure $F^\Sigma VA = \big((\tilde{A}_s)_s,\ (\sigma^{\tilde{A}})_\sigma,\ (R^{\tilde{A}})_R\big)$ satisfies all implications $J \in \mathcal{J}$. Hence let $J \in \mathcal{J}$ be as in (*) and $\bar{a}$ an assignment to the variables $x_1, \ldots, x_l$ in the algebra $F^\Sigma VA$ such that for $j = 1, \ldots, k$

$$\tilde{b}_j :\ = \big((t_1^j)^{\tilde{A}}(\bar{a}), \ldots, (t_{n_j}^j)^{\tilde{A}}(\bar{a})\big) \in R_j^{\tilde{A}} = \mathrm{im}\langle\ {}^{R_j}\pi_i^A\rangle.$$

Then there exist uniquely determined elements (recall that the maps $\langle\ {}^{R_j}\pi_i^A\rangle$ are monic) $b_j \in R_j^A$ $(j = 1, \ldots, k)$ with

$${}^{R_j}\pi_{i_j}^A(b_j) = (t_{i_j}^j)^A(\bar{a}) \text{ for all } j, i_j.$$

Since A is an $(\Omega^{\mathsf{H}}, \mathsf{def}, \mathcal{E}^{\mathsf{H}})$–model this means in particular: $(b_1, \ldots, b_k, \bar{a}) \in \mathsf{dom}\tilde{J}^A$, $\tilde{J}^A(b_1, \ldots, b_k, \bar{a}) \in Q^A$, and ${}^Q\pi_i^A \tilde{J}^A(b_1, \ldots, b_k, \bar{a}) = t_i^A(\bar{a})$ for $i = 1, \ldots, m$. It follows $\big(t_1^{\tilde{A}}(\bar{a}), \ldots, t_m^{\tilde{A}}(\bar{a})\big) = \big(t_1^A(\bar{a}), \ldots, t_m^A(\bar{a})\big) = \langle\ {}^Q\pi_i^A\rangle\big(\tilde{J}^A(b_1, \ldots, b_k, \bar{a})\big) \in Q^{\tilde{A}}$. $\diamond$

References

[1] J. Adámek, H. Herrlich, and J. Rosický. Essentially equational categories. *Cahiers Topol. Géom. Différentielles Catégoriques*, 24:175–192.

[2] J. Adámek and J. Rosicky. *Locally presentable and accessible categories.* Cambridge, 1994.

[3] M. Makkai and R. Paré. Accessible Categories. *Contemp. Math.*, volume 104. Amer. Math. Soc., Providence R.I., 1989.

[4] T. Mossakowski. Equivalences among various logical framework of partial algebras. In *Lecture Notes in Computer Science*, volume 1092, pages 403–433, 1996.

[5] H. -E. Porst. The algebraic theory of order. *Appl. Categorical Structures*, 1:423–440, 1993.

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