Algebraic lattices and locally finitely presentable categories

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Abstract. We show that subobjects and quotients respectively of any object K in a locally finitely presentable category form an algebraic lattice. The same holds for the internal equivalence relations on K. In fact, these results turn out to be—at least in the case of subobjects—nothing but simple consequences of well known closure properties of the classes of locally finitely presentable categories and accessible categories respectively. We thus get a completely categorical explanation of the well known fact that the subobject- and congruence lattices of algebras in finitary varieties are algebraic. Moreover we also obtain new natural examples: in particular, for any (not necessarily finitary) polynomial set-functor F, the subcoalgebras of an F-coalgebra form an algebraic lattice; the same holds for the lattices of regular congruences and quotients of these F-coalgebras.

Introduction

The notions of algebraic lattice and locally finitely presentable category respectively are of seemingly unrelated origins; while the former stems from studying subalgebra and congruence lattices of algebras in a (Birkhoff) variety, the latter is the result of axiomatizing categories which can be considered to be categories of algebras. In some more detail: (a) The class of locally finitely presentable categories is the class of all many-sorted essentially algebraic categories including finitary varieties, finitary quasivarieties, finitary Horn classes, and also categories like **Cat**, the category of small categories not belonging to any of the previous ones. (b) Algebraic lattices are precisely the lattices of subalgebras (equivalently of congruences) of algebras in a finitary variety.

It is well known that locally finitely presentable categories can also be considered as direct generalizations of algebraic lattices in the sense that a partially ordered set I is an algebraic lattice if and only if I — considered as a category $\mathcal{I}$ — is locally finitely presentable (see e.g. [5]). The compact elements in I then are the finitely presentable ones in $\mathcal{I}$.

The result mentioned above, that the subobjects of an algebra in a finitary variety form an algebraic lattice, might thus be rephrased as follows: for certain locally finitely presentable categories $\mathcal{K}$ (namely the finitary varieties) it holds that the categories of subobjects of objects in $\mathcal{K}$ are locally finitely presentable again. And this brings into question whether the restriction to finitary varieties is necessary.

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Similarly for congruences, where however, when generalizing from varieties, two different questions come up, since congruences in the algebraic sense might mean in general “kernel pair” or “internal equivalence relation”.

It is the purpose of this note to show that in fact all of these questions have an affirmative answer: We show that the class of locally finitely presentable categories, which is known to enjoy quite a number of useful closure properties, is also stable with respect to forming natural subcategories of their slice and co-slice categories, for example those of (plain) subobjects and regular quotients respectively. And these, being preordered sets, then have locally finitely presentable skeletons which are posets; due to the statement above they are even algebraic lattices. The case of regular quotients then also settles the question concerning kernel pairs—the respective categories are equivalent—while internal equivalence relations need an additional argument.

This way we not only provide a vast amount of new contexts in which algebraic lattices naturally occur—amongst others the lattices of subcoalgebras of and regular congruences on F-coalgebras for (not necessarily finitary) polynomial set-functors, thus in particular those of automata—but, maybe more importantly, we get a completely categorical explanation of why they appear in their original context.

1. Subobjects and subobject lattices

Fix a class $\mathcal{M}$ of monomorphisms in the category $\mathcal{K}$. By

$$\mathrm{Sub}_{\mathcal{M}}(K) = \{m\colon L \to K \mid L \in \mathsf{obj}\mathcal{K}, m \in \mathcal{M}\}$$

we denote the class of $\mathcal{M}$–subobjects of an object K of $\mathcal{K}$. $\mathrm{Sub}_{\mathcal{M}}(K)$ carries a preorder

$$m \leq m' \iff \exists f \in \mathcal{K}(L, L')\colon m = m' \circ f$$

and an equivalence relation

$$m \sim m' \iff m \leq m' \;\wedge\; m' \leq m$$

which can, alternatively, be described by

$$m \sim m' \iff \exists f \in \mathcal{K}(L, L'),\ f \text{ an isomorphism}\colon m = m' \circ f.$$

$\overline{\mathrm{Sub}_{\mathcal{M}}}(K)$, the class of equivalence classes of this relation, is partially ordered and can, in case $\mathcal{K}$ is a concrete category with a faithful monomorphism-preserving forgetful functor $|-|\colon \mathcal{K} \longrightarrow \mathbf{Set}$ be identified with the poset of $\mathcal{M}$–subobjects of K in the more traditional sense, namely as that of those $m\colon L \to K \in \mathcal{M}$ with $|m|$ a subset inclusion.

Recall that $\mathcal{M}$ is said to be *closed under intersections* provided that, whenever $(m_i\colon S_i \to K)_{i\in I}$ is a family of monomorphisms from $\mathrm{Sub}_{\mathcal{M}}(K)$ and $m\colon S \to K$ is its intersection in $\mathcal{K}$, then m is in $\mathrm{Sub}_{\mathcal{M}}(K)$. The class of all monomorphisms is trivially closed under intersections.

$\mathrm{Sub}_{\mathcal{M}}(K)$ will, in the sequel, be considered as a full subcategory of the slice (comma) category $\mathcal{K}{\downarrow}K$. Occasionally we denote an object $f\colon L \to K$ in $\mathcal{K}{\downarrow}K$ by

(L, f). $\overline{\mathrm{Sub}_{\mathcal{M}}}(K)$, considered as a category, is then the skeleton of $\mathrm{Sub}_{\mathcal{M}}(K)$ since, for $m\colon L \longrightarrow K,\ m'\colon L' \longrightarrow K \in \mathcal{M}$ one has

$$m \leq m' \Longleftrightarrow \exists\ f\colon (L, m) \longrightarrow (L', m') \text{ in } \mathcal{K} \downarrow K.$$

We will, in the sequel, restrict ourselves to the following choices of $\mathcal{M}$:

$\mathcal{M}_m$**:** the class of all monomorphisms in $\mathcal{K}$;
$\mathcal{M}_{str}$**:** the class of all strong monomorphisms in $\mathcal{K}$;
$\mathcal{M}_{reg}$**:** the class of all regular monomorphisms in $\mathcal{K}$.

This is motivated by the fact that we will work in locally finitely presentable categories $\mathcal{K}$ only, where plain and strong monomorphisms are always part of an existing factorization structure for morphisms (see [5, 1.61]) and regular monomorphisms occasionally are.

Recall in this context the following

Fact 1 (see e.g. [2])**.** *Let $\mathcal{K}$ be a category; then*

1. $\mathcal{M}_{reg} \subset \mathcal{M}_{str} \subset \mathcal{M}_m$*;*
2. $\mathcal{M}_m$ *and* $\mathcal{M}_{str}$ *are closed under intersections;*
3. $\mathcal{M}_{reg}$ *is closed under intersections, provided that $\mathcal{K}$ has products.*

Writing $\mathrm{Sub}_x(K)$ instead of $\mathrm{Sub}_{\mathcal{M}_x}$ for any specified class $\mathcal{M}_x$ of monomorphisms we have the following simple result:

Proposition 2. *Let $\mathcal{K}$ be a category and K a $\mathcal{K}$–object.*

1. *For the chain of full subcategories*
$$\mathrm{Sub}_{reg}(K) \subset \mathrm{Sub}_{str}(K) \subset \mathrm{Sub}_m(K) \subset \mathcal{K}{\downarrow}K$$
it holds that each of these categories is closed in its immediate supercategory (and thus in all of its supercategories) with respect to limits, provided that $\mathcal{K}$ has products.
2. *Similarly, in the chain of full subcategories*
$$\mathrm{Sub}_{reg}(K) \subset \mathrm{Sub}_m(K) \subset \mathcal{K}{\downarrow}K$$
each of these categories is closed in its immediate supercategory (and thus in all of its supercategories) with respect to directed colimits, provided that $\mathcal{K}$ is a locally presentable category.

Proof. Concerning limits we only have to consider products since, in each of the subcategories mentioned above, equalizers are identities. But products in $\mathcal{K} \downarrow K$ are multiple pullbacks (over K) such that, for each class $\mathcal{M}$ of monomorphism, products of objects from $\mathrm{Sub}_{\mathcal{M}}(K)$ in $\mathcal{K}{\downarrow}K$ are intersections of subobjects. Thus the claimed closure properties with respect to products follows from Fact 1 above.

Concerning directed colimits recall that the forgetful functor $\mathcal{K} \downarrow K \to \mathcal{K}$ creates these; it therefore suffices to observe that the classes $\mathcal{M}_m$ and $\mathcal{M}_{reg}$ are stable under directed colimits (see [5, 1.60]). □

Remark 3. Strong monomorphisms in a locally presentable category might fail to be stable under directed colimits as is shown in [1].

For convenience we recall the following closure property of the class of locally presentable categories.

Fact 4. *A full subcategory of a locally finitely presentable category is locally finitely presentable again, if it is closed under limits and directed colimits* ([5, 2.48]).

From here we easily arrive at

Theorem 5. *Let $\mathcal{K}$ be a locally finitely presentable category and K a $\mathcal{K}$–object. Then the categories $\mathrm{Sub}_m(K)$ and $\mathrm{Sub}_{reg}(K)$ are locally finitely presentable and, equivalently, the partially ordered sets $\overline{\mathrm{Sub}_m}(K)$ and $\overline{\mathrm{Sub}_{reg}}(K)$ are algebraic lattices.*

Proof. By [5, 1.57] $\mathcal{K} \downarrow K$ is locally finitely presentable, so that (see 4 above) it suffices to show that $\mathrm{Sub}_m(K)$ and $\mathrm{Sub}_{reg}(K)$ are closed in $\mathcal{K} \downarrow K$ w.r.t. limits and directed colimits. But this is clear by the preceding proposition.

Finally, use that facts that the skeleton of a finitely locally presentable category $\mathcal{K}$, being equivalent to $\mathcal{K}$, clearly is locally finitely presentable again and that, moreover, locally presentable categories are wellpowered. □

Examples 6.
(1) *For any (Birkhoff) variety $\mathcal{V}$ the subalgebras of each algebra in $\mathcal{V}$ form an algebraic lattice.*
This is the classical example of which we now were able to give a purely categorical proof. (Varieties are locally finitely presentable with subalgebras presented by monomorphisms.)
(2) *Let Σ be a finitary S-sorted relational signature. Then the subobjects of each relational structure of type Σ form an algebraic lattice.*
The transition systems of [9] (a very restricted type of transition systems), which are just relational systems with one binary relation, are a particular instance. (The category $\mathbf{Rel}\Sigma$ of the relational structures in question is locally finitely presentable (see [5]). The subobjects (S-sorted subsets) are given by monomorphisms as is easily seen.)
(3) *$F\colon \mathbf{Set} \to \mathbf{Set}$ be a finitary functor. Then the subalgebras of each F-algebra $(A, \alpha\colon FA \to A)$ form an algebraic lattice.*
(The category $\mathbf{Alg}(F)$ of F-algebras in question is locally finitely presentable and subalgebras are represented by monomorphisms.)
(4) *Let $F\colon \mathbf{Set} \to \mathbf{Set}$ be a (not necessarily finitary) polynomial functor. Then the subcoalgebras of each F-coalgebra $(A, \alpha\colon A \to FA)$ form an algebraic lattice.*
A particular instance of this situation is the category of automata, since automata with input set I and output set O are the coalgebras for the (polynomial) functor $X \mapsto O \times X^I$. (The category $\mathbf{Coalg}(F)$ of F-coalgebras in question is locally finitely presentable by [4]. The subcoalgebras are precisely the regular subobjects in $\mathbf{Coalg}(F)$ (see [3]). Now use Theorem 5.)

Remark 7. In the examples (3) and (4) above the sub(co)algebra lattices are not only algebraic lattices but even *algebraic sublattices* of $\mathcal{P}(A)$, that is, they are closed in $\mathcal{P}(A)$ under intersections and directed unions. (Not every algebraic lattice embeddable into $\mathcal{P}(A)$ is an algebraic sublattice of the latter, as the example $\mathcal{P}(B)$ for $B \subset A,\ B \neq A$ shows.) This gives rise to consider the following questions.

First, Example 6(4) above can be generalized. If F, more generally, is a functor on **Set** which preserves intersections, then it is easily seen that the lattice of subcoalgebras of any F-coalgebra (A, α) is closed in $\mathcal{P}(A)$ under intersections and directed unions and thus an algebraic lattice, more precisely, an algebraic sublattice of the powerset lattice $\mathcal{P}(A)$. This property characterizes intersection preserving **Set** functors:

Let $F\colon \mathbf{Set} \to \mathbf{Set}$ be a functor such that the subcoalgebras of each F-coalgebra $(A, \alpha\colon A \to FA)$ form an algebraic sublattice of $\mathcal{P}(A)$. Then F preserves intersections.

Indeed, assume there is a set A and a family of subsets $(A_i)_{i\in I}$ with F not preserving its intersection, that is, there exists some $x \in A$ with

$$\forall_{i\in I} x \in F(A_i) \quad \text{and} \quad x \notin F(\bigcap_{i\in I} A_i).$$

The coalgebra (A, α) with $\alpha(a) = x$ for each $a \in A$ then has as (the underlying sets of) its subcoalgebras precisely those subsets $X \subset A$ with $x \in F(X)$. In particular each A_i is a subcoalgebra of (A, α), but not $\bigcap_{i\in I} A_i)$. Thus, the lattice of subcoalgebras of (A, α) fails to be an algebraic sublattice of $\mathcal{P}(A)$.

In view of this and Example (3) above, where the subalgebras of an F-algebra (A, α) also form an algebraic sublattice of $\mathcal{P}(A)$, one similarly might ask whether this property characterizes finitary set functors. We didn't succeed in solving this problem completely. We only know the following: The (non-finitary) covariant powerset functor $\mathcal{P}$ has algebras $(A, \alpha\colon \mathcal{P}(A) \to A)$ such that

- its subalgebras form an algebraic sublattice of $\mathcal{P}(A)$. The $\mathcal{P}$-algebra with $\alpha(U) = x_0$ for all $U \subset A$ and $x_0 \in A$ fixed is an example. Its subalgebras are precisely the subsets U with $x_0 \in U$.

but also algebras such that

- its subalgebras do not form an algebraic sublattice of $\mathcal{P}(A)$. Put $A = \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $\alpha(U) = \min U$ for all $U \neq \bar{\mathbb{N}} \setminus \{0\}$ and $\alpha(\bar{\mathbb{N}} \setminus \{0\}) = 0$. The subalgebras of (A, α) are the subsets of A containing $\infty = \min \emptyset$ which are different from $\bar{\mathbb{N}} \setminus \{0\}$, and these do not form an algebraic sublattice of $\mathcal{P}(A)$ (consider the directed join of all finite subsets $F \subset \bar{\mathbb{N}}$ with $\infty \in F$, $0 \notin F$).

2. Quotients and quotient lattices

Dually to the case of subobjects we might fix a class $\mathcal{E}$ of epimorphisms in $\mathcal{K}$ and form the class

$$\mathrm{Quot}_{\mathcal{E}}(K) = \{e\colon K \longrightarrow L \mid L \in \mathsf{obj}\mathcal{K},\ e \in \mathcal{E}\}$$

of $\mathcal{E}$–quotients of a $\mathcal{K}$–object K and consider this as a full subcategory of $K\downarrow\mathcal{K} = (\mathcal{K}^{\mathsf{op}}\downarrow K)^{\mathsf{op}}$. $\mathrm{Quot}_{\mathcal{E}}(K)$ then is preordered by

$$e \leq e' \Longleftrightarrow \exists f\colon \mathcal{K}(L', L)\colon f \circ e' = e$$

and $\overline{\mathrm{Quot}_{\mathcal{E}}}(K)$, the class of equivalence classes of the equivalence relation

$$e \sim e' \Longleftrightarrow e \leq e' \wedge e' \leq e \Longleftrightarrow \exists f \in \mathcal{K}(L', L),\ f \text{ an isomorphism}\colon f \circ e' = e$$

is partially ordered and, considered as a category, the skeleton of $\mathrm{Quot}_{\mathcal{E}}(K)$.

For the same reasons as above we concentrate—since $\mathcal{K}$ is assumed from now on to be locally finitely presentable—on the cases

- $\mathcal{E}_e$ **:** the class of all epimorphisms in $\mathcal{K}$;
- $\mathcal{E}_{str}$**:** the class of all strong epimorphims in $\mathcal{K}$, coinciding with the class of all extremal epimorphisms;
- $\mathcal{E}_{reg}$**:** the class of all regular epimorphisms in $\mathcal{K}$.

Note that, for each $i \in \{e, str, reg\}$, $\overline{\mathrm{Quot}_{\mathcal{E}_i}}(K)$ is a complete lattice by duality. As for subobjects we write $\mathrm{Quot}_i(K)$ instead of $\mathrm{Quot}_{\mathcal{E}_i}(K)$.

One cannot proceed, however, in analogy to the proof of Theorem 5, in order to obtain results analogous to those of that Theorem for the various kinds of quotients: Though $K\downarrow\mathcal{K}$ is locally finitely presentable (see again [5, 1.57]), $\mathrm{Quot}_e(K)$ will not be closed in $K\downarrow\mathcal{K}$ under products as already the most simple case of $\mathcal{K} = \mathbf{Set}$ shows.

One thus is forced to resort to the definition of a locally finitely presentable category. Since, by duality, all the categories Quot_i are cocomplete, it suffices to investigate the problem whether or not one can find suitable sets of finitely presentable objects in the categories of strong and regular quotients respectively.

For the case of strong quotients this is done in [1]. We cite their result as follows.

Lemma 8 ([1])**.** *Let $\mathcal{K}$ be a locally finitely presentable category and K a $\mathcal{K}$-object. Then every strong epimorphism $f\colon K \to L$ in $\mathcal{K}$ is a filtered colimit (in $K\downarrow\mathcal{K}$) of strong epimorphisms, which are finitely presentable as $K\downarrow\mathcal{K}$-objects.*

For identifying a suitable set of finitely presentable objects in $\mathrm{Quot}_{reg}(K)$ we need the following lemma, which is probably well known; we include a proof since we couldn't find a reference.

Lemma 9. *Let $\mathcal{K}$ be a category and K a $\mathcal{K}$–object. Then the forgetful functor $U\colon K\downarrow\mathcal{K} \longrightarrow \mathcal{K}$ creates connected colimits, thus filtered and directed colimits in particular.*

Proof. Let $D\colon I \longrightarrow K\downarrow\mathcal{K}$ be a connected diagram, $D(i) = (q_i, D_i)$, and $\lambda_i\colon D_i \longrightarrow L$ a colimit of $U \circ D$. If $d_{ik}\colon D_i \longrightarrow D_k$ is a connecting morphism in that diagram, then (since d_{ik} is a morphism in $K\downarrow\mathcal{K}$)

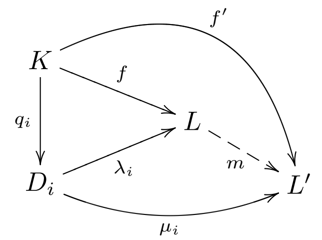
$$\lambda_i \circ q_i = \lambda_k \circ d_{ik} \circ q_i = \lambda_k q_k = \lambda_j \circ q_j.$$

It follows that, for all $i, j \in I$, $\lambda_i \circ q_i = \lambda_j \circ j =: f$. Thus, (f, L) is a $K\downarrow\mathcal{K}$–object such that all colimiting morphisms are $K\downarrow\mathcal{K}$–morphisms $\lambda_i\colon (q_i, D_i) \longrightarrow (f, L)$ and (f, L) is the only such object. To show that this is actually a directed colimit of D

let $\mu_i\colon (q_i, D_i) \longrightarrow (f', L')$ be a compatible cocone for D. Since $\mu_i\colon D_i \longrightarrow L'$ is a compatible cocone for $U \circ D$ there exists a unique $\mathcal{K}$–morphism $m\colon L \longrightarrow L'$ with $m \circ \lambda_i = \mu_i$ for each i. m is a morphism $(f, L) \longrightarrow (f', L')$ in $K{\downarrow}\mathcal{K}$ since, for each i,

$$f' = \mu_i \circ q_i = m \circ \lambda_i \circ q_i = m \circ f.$$

The following diagram depicts the situation.



□

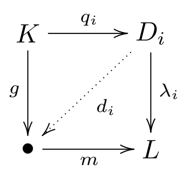
We also have, similarly to the case of subobjects:

Proposition 10. *Let $\mathcal{K}$ be a locally presentable category and K a $\mathcal{K}$–object. Then in the chain of full subcategories*

$$\mathrm{Quot}_{reg}(K) \subset \mathrm{Quot}_{str}(K) \subset \mathrm{Quot}_{e}(K) \subset K{\downarrow}\mathcal{K}$$

each of these categories is closed in its immediate supercategory (and thus in all of its supercategories) with respect to directed colimits.

Proof. We need to show that the morphism $f\colon K \longrightarrow L$ as constructed in the proof of Lemma 9 belongs to $\mathcal{E}_j$ wherever all $q_i \in \mathcal{E}_j$. For $\mathcal{E}_e$ this is clear since the family (λ_i) is jointly epimorphic. For $\mathcal{E}_{reg}$ it is equally obvious since (directed) colimits and coequalizers commute. For $\mathcal{E}_{str}$ we need to show that, given any factorization $f = m \circ g$ with m a monomorphism, m is an isomorphism. Now, in each of the diagrams



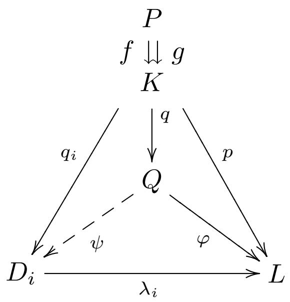
there exists a unique diagonal d_i making the triangles commute since, by our assumption, q_i is a strong epimorphism. Since the colimiting cone (λ_i) is an extremal epi–cone, m must be an isomorphism. □

Proposition 11. *Let $f, g\colon P \longrightarrow K$ be a pair of morphisms in $\mathcal{K}$ where P is finitely presentable. If $q\colon K \longrightarrow Q$ is a coequalizer of (f, g) then (q, Q) is finitely presentable in $K{\downarrow}\mathcal{K}$. Consequently, (q, Q) then is finitely presentable in* $\mathrm{Quot}_{\mathcal{E}}(K)$ *provided that* $\mathrm{Quot}_{\mathcal{E}}(K)$ *is closed in $K{\downarrow}\mathcal{K}$ under directed colimits.*

Proof. Let $(q_i, D_i) \xrightarrow{\lambda_i} (p, L)$ be a directed colimit in $K{\downarrow}\mathcal{K}$ and $\varphi\colon (q, Q) \longrightarrow (p, L)$ a morphism. We need to find an (essentially) unique factorization of φ through some (q_i, D_i).

Since $\lambda_i\colon D_i \longrightarrow L$ is a directed colimit in $\mathcal{K}$ und P is finitely presentable, the $\mathcal{K}$-morphism $pf = pg$ factors essentially uniquely through some D_{i_0} as

$$pf = pg = \lambda_{i_0} \circ h_{i_0}.$$



Since $\lambda_{i_0} q_{i_0} f = pf = \varphi q f = \varphi q g = \lambda_{i_0} q_{i_0} g$, too, there is some $i_1 \geq i_0$ such that

$$D(i_0 \longrightarrow i_1) q_{i_0} f = D(i_0 \longrightarrow i_1) q_{i_0} g$$

which implies $q_{i_1} \circ f = q_{i_1} \circ g$ (since D is a diagram in $K \downarrow \mathcal{K}$). Thus there exists a unique $\psi\colon Q \longrightarrow D_{i_1}$ with $q_{i_1} = \psi \circ q$ making ψ even a $K{\downarrow}\mathcal{K}$–morphism $\psi\colon (q, Q) \longrightarrow (q_{i_1}, D_{i_1})$. Since $\lambda_{i_1} \psi g = \lambda_{i_1} q_{i_1} = p = \varphi q$ and q is an epimorphism we conclude $\lambda_{i_1} \circ \psi = \varphi$ as required. □

Theorem 12. *Let $\mathcal{K}$ be a locally finitely presentable category and K an object in $\mathcal{K}$. Then the categories* $\mathrm{Quot}_{str}(K)$ *and* $\mathrm{Quot}_{reg}(K)$ *are locally finitely presentable. Equivalently, the partially ordered sets* $\overline{\mathrm{Quot}_{str}}(K)$ *and* $\overline{\mathrm{Quot}_{reg}}(K)$ *of strong and regular quotients respectively of K in $\mathcal{K}$ are algebraic lattices.*

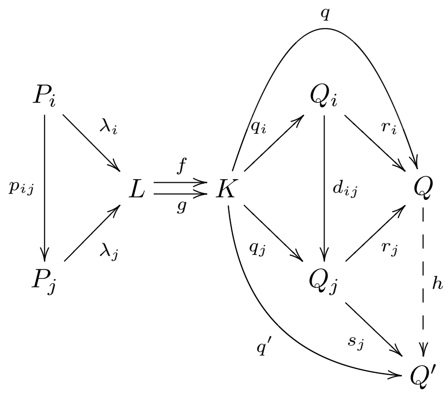
Proof. Since $\mathrm{Quot}_{str}(K)$ and $\mathrm{Quot}_{reg}(K)$ are cocomplete and $\mathcal{K}$ is co-wellpowered (see [5, 1.58])), it follows from Lemma 8 in connection with Proposition 10 that $\mathrm{Quot}_{str}(K)$ is locally finitely presentable.

The finitely presentables in $\mathrm{Quot}_{reg}(K)$ described in the previous proposition then also form a set and it remains to show that every (q, Q) in $\mathrm{Quot}_{reg}(K)$ is a directed colimit of such objects. Thus, let (q, Q) be a $\mathrm{Quot}_{reg}(K)$–object and

$$L \underset{g}{\overset{f}{\rightrightarrows}} K \xrightarrow{q} Q$$

a coequalizer diagram in $\mathcal{K}$. Let $P_i \xrightarrow{\lambda_i} L$ $(i \in I)$ be a directed colimit with all P_i finitely presentable in $\mathcal{K}$. Put, for each i, $f_i = f \circ \lambda_i\colon P_i \longrightarrow K$, $g_i = g \circ \lambda_i\colon P_i \longrightarrow K$ and let $q_i\colon K \longrightarrow Q_i$ be a coequalizer of (f_i, g_i). There results a directed diagram $Q_i \xrightarrow{d_{ij}} Q_j$ $(i \leq j)$ with $d_{ij} \circ q_i = q_j$ and a (unique) compatible cocone $r_i\colon Q_i \longrightarrow Q$ with $r_i \circ q_i = q$. In particular, the d_{ij} and r_i are all morphisms in $\mathrm{Quot}_{reg}(K)$.

If now $s_j\colon (q_j, K) \longrightarrow (q', K)$ is a compatible cocone, $s_j \circ q_j = q'$ follows for all j and thus $q' f\lambda_i = s_i q_i f\lambda_i = s_i q_i g\lambda_i = q' g\lambda_i$ for all $i \in I$. Hence $q'f = q'g$ and there is a unique $h\colon Q \longrightarrow Q'$ with $hq = q'$, implying that h is a $\mathrm{Quot}_{reg}(K)$–morphism in particular. The following diagram illustrates the situation.



It is now easy to see that h is also unique with respect to $hr_j = s_j$ for all j. □

As examples we mention

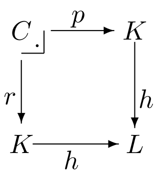
Examples 13. (1) *In any (Birkhoff) variety $\mathcal{V}$ the quotients of an algebra form an algebraic lattice.*
This is the most familiar case, usually presented in the form of Corollary 17 below. (It follows from the theorem above since in a variety the usual quotients are precisely the regular ones.)

(2) *Let $F\colon$ **Set** $\to$ **Set** be a (not necessarily finitary) polynomial functor. Then the quotients of each F-coalgebra $(A, \alpha\colon A \to FA)$ form an algebraic lattice.*
As in the case of subobjects this corollary includes the case of automata. (**Coalg**(F) is a locally finitely presentable category as already mentioned above. The quotients in **Coalg**(F) are, as in any category of coalgebras, the epimorphisms; if F is polynomial, these agree with the regular epimorphims since, in this case, the forgetful functor not only creates coequalizers but also kernel pairs (polynomial set-functors preserve kernel pairs as is easily seen). Now use the theorem above.)

Remark 14. The category $\mathrm{Quot}_e(K)$ of epimorphic quotients of an object K in a locally finitely presentable category $\mathcal{K}$ in general fails to be locally finitely presentable, which is not really surprising (looking, e.g., at epimorphisms in the category of unital rings). Indeed, one cannot find, in general, a set of epimorphic quotients of K, each finitely presentable in $K{\downarrow}\mathcal{K}$, such that every f in $\mathrm{Quot}_e(K)$ can be represented as a filtered colimit (in $K{\downarrow}\mathcal{K}$) of those, as is demonstrated by Example 1.5(b) of [7] (see also [1, 2.11]). Using Proposition 10 the claim follows.

3. Congruences and congruence lattices

Recall that a *kernel pair* on an object K in a category $\mathcal{K}$ is a pullback of the form

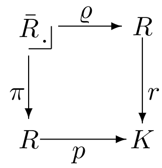


Kernel pairs are special instances of *internal equivalence relations* on K in the category $\mathcal{K}$, that is, of jointly monomorphic pairs $p, r\colon R \to K$ (or equivalently, in the presence of products, subobjects of $K \times K$) which are

reflexive: that is, the diagonal Δ_K of K factors through $\langle p, r\rangle$ in $\mathrm{Sub}_m(K \times K)$ or, equivalently, $\Delta_K \leq \langle p, r\rangle$ in $\mathrm{Sub}_m(K \times K)$

symmetric: that is, the subobjects $\langle p, r\rangle$ and $\langle r, p\rangle$ are isomorphic in $\mathrm{Sub}_m(K \times K)$,

transitive: that is, given a pullback



then one has $\langle p \circ \pi, r \circ \rho\rangle \leq \langle p, r\rangle$ in $\mathrm{Sub}_m(K \times K)$, that is, there exists a morphism $\tau\colon \bar{R} \to R$ with

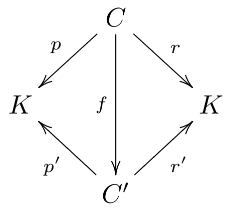
$$r \circ \tau = r \circ \rho \quad \text{and} \quad p \circ \tau = p \circ \pi \tag{1}$$

It is a special feature of a (Birkhoff) variety $\mathcal{V}$ that kernel pairs and internal equivalence relations in $\mathcal{V}$ coincide. This is why in universal algebra the term *congruence relation* is used simultaneously for both. The abelian group $G_n :=$ $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid n \mid x - y\}$ ($n \in \mathbb{N}$), considered as a subgroup of $\mathbb{Z} \times \mathbb{Z}$, together with the projections $G_n \longrightarrow \mathbb{Z}$, for example is a kernel pair on $\mathbb{Z}$ in **Ab**, the category of Abelian groups, but not so in **TorfAb**, the category of torsion free Abelian groups (which is not a variety), where it is only an internal equivalence relation!

The observation that the congruences of an algebra A in a variety $\mathcal{V}$ form an algebraic lattice therefore leads to two different questions when generalizing from varieties to arbitrary locally finitely presentable categories:

- Do the internal equivalence relations on an object K form an algebraic lattice?
- Do the kernel pairs on an object K form an algebraic lattice?

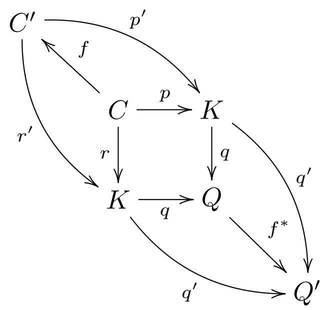
We will answer both questions in the style of the previous sections by showing that both, the category $\mathrm{Kerp}(K)$ of kernel pairs on K and the category $\mathrm{EqRel}(K)$ of internal equivalence relations on K are locally finitely presentable, where both categories are understood to be full subcategories of of $\mathrm{Sub}_m(K \times K)$. Note in particular that this means that a morphism $f\colon (C,p,r) \longrightarrow (C',p',r')$ between kernel pairs and between internal equivalence relations respectively is a K–morphism $f\colon C \longrightarrow C'$ such that the following diagram commutes. Such an f is (in both cases) necessarily a $\mathcal{K}$–monomorphism.



Again, the skeletons of theses categories are partially ordered sets and can be identified with sets of equivalence relations on the underlying set of K, provided that $\mathcal{K}$ has a faithful functor into **Set** which preserves limits.

The following is a well known fact:

Fact 15. *If $\mathcal{K}$ is a category with pullbacks and coequalizers, then the categories* $\mathrm{Kerp}(K)$ *and* $\mathrm{Quot}_{reg}(K)$ *are equivalent for each K in $\mathcal{K}$. This equivalence is given by assigning to the kernel pair (C,p,r) on K a (chosen) coequalizer $q\colon K \longrightarrow Q$ of (p,r) and to $f\colon (C,p,r) \longrightarrow (C',p',r')$ the unique morphism $f^*\colon (q,Q) \longrightarrow (q',Q')$ with $f^* \circ q = q'$ resulting from the coequalizer property of q.*



Thus, the following results are nothing but reformulations of (a part of) Theorem 12 and Example 13 (1) respectively.

Theorem 16. *Let $\mathcal{K}$ be a locally finitely presentable category and K a $\mathcal{K}$–object. Then* $\mathrm{Kerp}(K)$*, the category of kernel pairs on K, is a locally finitely presentable category. Equivalently,* $\overline{\mathrm{Kerp}}(K)$*, the skeleton of* $\mathrm{Kerp}(K)$*, is an algebraic lattice.*

Corollary 17. *Let A be an algebra in a finitary variety $\mathcal{V}$. Then the congruences on A (in the algebraic sense) form an algebraic lattice.*

Kernels of homorphisms in $\mathbf{Coalg}(F)$ are also called "congruences" (see [6]); the congruences on a F-coalgebra $(A, \alpha\colon A \to FA)$ (over **Set**) can be characterized as equivalence relations Θ on A which are contained in the kernel congruence of $A \xrightarrow{\alpha} FA \xrightarrow{Fq} F(A/\Theta)$. Note that, in general, these are not kernel pairs. If, however, F is a polynomial **Set** functor, then they are (see the argument in Example

13 (2)), and they are precisely the *regular congruences* in the sense of [6]. Since **Coalg**(F) has pullbacks and coequalizers (which are created by the underlying functor) we now also have, maybe surprisingly, the following result, which again is an equivalent formulation of Example 13 (2) above.

Corollary 18. *Let* $F\colon \mathbf{Set} \to \mathbf{Set}$ *be a (not necessarily finitary) polynomial functor. Then the regular congruences on each* F*-coalgebra* $(A, \alpha\colon A \to FA)$ *form an algebraic lattice.*

Concerning internal equivalence relations we also have the following generalization of the classical result formulated in Corollary 17 above.

Theorem 19. *Let* $\mathcal{K}$ *be a locally finitely presentable category and* K *a* $\mathcal{K}$*–object. Then* $\mathrm{EqRel}(K)$*, the category of internal equivalence relations on* K*, is a locally finitely presentable category. Equivalently,* $\overline{\mathrm{EqRel}}(K)$*, the skeleton of* $\mathrm{EqRel}(K)$*, is an algebraic lattice.*

Proof. Since $\mathrm{Sub}_m(K \times K)$ is a locally finitely presentable category by Theorem 5 we only need to show that $\mathrm{EqRel}(K)$ is closed in $\mathrm{Sub}_m(K \times K)$ under products, i.e., intersections in $\mathcal{K}$, and under directed colimits.

Symmetry clearly is stable under limits and directed colimits since limits (and colimits respectively) of isomorphic diagrams are isomorphic.

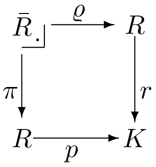
Reflexivity is stable under (directed) colimits and intersections: assume that, for each $i \in I$, there exists a morphism $f_i\colon (K, \Delta) \to (R_i, \langle p_i, r_i\rangle)$ and that $(R, \langle p, r\rangle)$ is an intersection of the $(R_i, \langle p_i, r_i\rangle)$; then, since each f_i necessarily is a monomorphism, the definition of intersection yields a morphism $f\colon (K, \Delta) \to (R, \langle p, r\rangle)$. The case of colimits is trivial: compose f_i with the colimit map!

Finally, let $(R_i, \langle p_i, r_i\rangle)$ $(i \in I)$ be a family of transitive relations with morphisms $\tau_i\colon \bar{R}_i \to R_i$ satisfying

$$r_i \circ \tau_i = r_i \circ \rho_i \quad \text{and} \quad p_i \circ \tau_i = p_i \circ \pi_i \tag{2}$$

(see the definition of transitivity above). Let $t_i\colon (R, \langle p, r\rangle) \to (R_i, \langle p_i, r_i\rangle)$ be an intersection of the given family and denote by $d_i\colon (R^*, \langle p^*, r^*\rangle) \to (\bar{R}_i, \langle p_i \circ \pi_i, r_i \circ \rho_i\rangle)$ an intersection of the corresponding $\langle p_i\pi_i, r_i\rho_i\rangle$. The situation is depicted in Diagrams 1. and 2. at the end of this proof.

Given a pullback



we need to find a morphism $\tau\colon \bar{R} \to R$ satisfying Equations (1). Since limits commute we can assume

$$(R^*, p^*, r^*) = (\bar{R}, p \circ \pi, r \circ \rho). \tag{3}$$

Let now τ be the morphism with $\tau_i \circ d_i = t_i \circ \tau$ given by intersection. Then

$$\begin{aligned} r \circ \rho = r^* &= r_i \circ \rho_i \circ d_i && \text{(by Equation (3) and Diagram 1)} \\ &= r_i \circ \tau_i \circ d_i && \text{(by Equations (2))} \\ &= r_i \circ t_i \circ \tau && \text{(by definition of } \tau\text{)} \\ &= r \circ \tau \end{aligned}$$

Finally, let $t_i \colon (R_i, \langle p_i, r_i \rangle) \to (R, \langle p, r \rangle)$ be a colimit of the given family—now assumed to be directed—and let $d_i \colon (\bar{R}_i, \langle p_i \circ \pi_i, r_i \circ \rho_i \rangle) \to (R^*, \langle p^*, r^* \rangle)$ be a colimit of the corresponding one (which is easily seen to be directed as well). The situation is depicted in the diagrams above if one reverts the direction of arrows labelled d_i and t_i. Again, since directed colimits commute with pullbacks in the locally finitely presentable category $\mathcal{K}$, we might assume that Equation (3) holds. The family of morphisms τ_i now induces some $\tau \colon \bar{R} = R^* \to R$ between the directed colimits such that the equation

$$\tau \circ d_i = t_i \circ \tau_i \tag{4}$$

holds, for each $i \in I$. Now

$$\begin{aligned} r \circ \rho \circ d_i = r^* \circ d_i &= r_i \circ \rho_i && \text{(by Equation (3) and Diagram 1)} \\ &= r_i \circ \tau_i && \text{(by Equations (2))} \\ &= r \circ t_i \circ \tau_i && \text{(by Diagram 2)} \\ &= r \circ \tau \circ d_i && \text{(by definition of } \tau\text{)} \end{aligned}$$

Since directed colimits are inherited from $\mathcal{K}$ we can cancel the family $(d_i)_I$ jointly and the proof is complete.

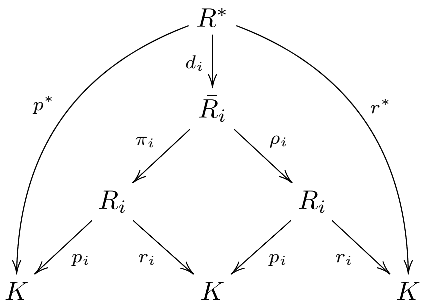


Diagram 1.

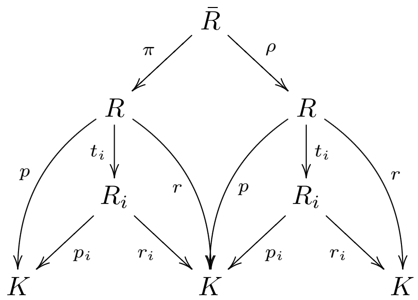


Diagram 2.

□

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