ON SUBCATEGORIES OF THE CATEGORY OF HOPF ALGEBRAS

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Abstract. The various canonical subcategories of the category $\mathbf{Hopf}_R$ of Hopf algebras over a commutative ring R, like those of (co)commutative Hopf algebras or Hopf algebras whose antipode is bijective or of order 2, are shown to be locally presentable categories and reflective and coreflective in their respective supercategories. The reflectivity results provided only hold for commutative von Neumann regular rings, while most of the coreflectivity results are valid over any ring. As a consequence one gets existence of free commutative Hopf algebras over coalgebras and cofree cocommutative Hopf algebras over algebras.

1. Introduction

It has been shown recently (see [5]) that the category $\mathbf{Hopf}_R$ of Hopf algebras over a commutative ring R is a locally presentable category, thus, in particular it has all limits and colimits, is wellpowered and cowellpowered and has a generator (see [1]). Also, coreflections of $\mathbf{Bialg}_R$, the category of bialgebras over R, into $\mathbf{Hopf}_R$ have been shown to exist for any such ring, and existence of reflections has been proved in case the ring is in addition von Neumann regular. In this restricted case also constructions of the reflections and coreflections respectively could be described, generalizing and dualizing respectively the familiar construction of the Hopf envelope of a bialgebra over a field k.

Note that the condition on R to be von Neumann regular is needed to ensure that homomorphisms of coalgebras and, thus, of bialgebras have image factorizations: this fact is crucial for the explicit description of limits of bialgebras given in [5]. That is what we refer to (possibly implicitly), whenever this condition is used in the sequel.

This note complements these results by investigating the categories

- ${}_{c}\mathbf{Hopf}_R$, the category of commutative Hopf algebras over R,
- ${}_{coc}\mathbf{Hopf}_R$, the category of cocommutative Hopf algebras over R,
- ${}_{c,coc}\mathbf{Hopf}_R$, the category of commutative and cocommutative Hopf algebras over R,
- ${}_{S^2=\mathsf{id}}\mathbf{Hopf}_R$, the category of Hopf algebras over R with antipode satisfying $S^2 = \mathsf{id}$,
- ${}_{bi}\mathbf{Hopf}_R$, the category of Hopf algebras over R with bijective antipode.

Recall that these subcategories are related as indicated in the digram below.

$$\begin{array}{ccccccc} & & {}_{c}\mathbf{Hopf}_R & & & & \\ & \nearrow & & \searrow & & & \\ {}_{c,coc}\mathbf{Hopf}_R & & & & {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \hookrightarrow {}_{bi}\mathbf{Hopf}_R \hookrightarrow \mathbf{Hopf}_R \\ & \searrow & & \nearrow & & & \\ & & {}_{coc}\mathbf{Hopf}_R & & & & \end{array}$$

Moreover, each of the categories ${}_{c,coc}\mathbf{Hopf}_R$, ${}_{coc}\mathbf{Hopf}_R$, ${}_{c}\mathbf{Hopf}_R$, and $\mathbf{Hopf}_R$ is a full subcategory of the respective subcategory of $\mathbf{Bialg}_R$, denoted analogously.

The following diagram then summarizes our results where each label c marks a coreflective embedding and each label r a reflective one. Labels in brackets indicate that the respective result only holds over von Neumann regular rings.

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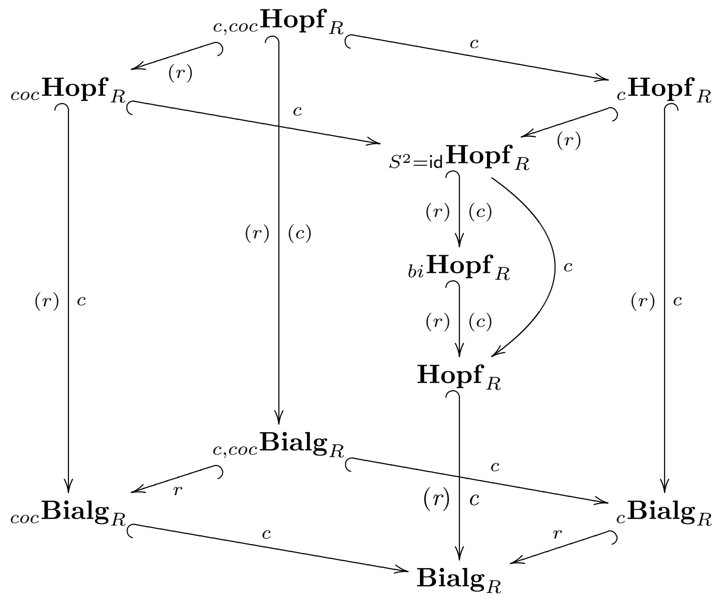

Note that, for $R = k$, a field, the reflection of $\mathbf{Bialg}_R$ into $\mathbf{Hopf}_R$ is the well known Hopf envelope of a bialgebra and the reflection of $\mathbf{Bialg}_R$ into ${}_{bi}\mathbf{Hopf}_R$ has been proved in [7].

Composition of adjunctions then yields the following additional results: (1) For any commutative ring there exists a cofree cocommutative Hopf algebra over any algebra. (2) For any von Neumann regular ring there exists a free commutative Hopf algebra over any coalgebra. The latter statement generalizes a result of Takeuchi (see [8]) for the case of fields.

2. Prerequisites

To reduce the problem we will make use of the following well known facts.

Fact 1. *For every symmetric monoidal category $\mathbb{C}$, the categories $\mathbf{Mon}\mathbb{C}$ of monoids in $\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$ of comonoids in $\mathbb{C}$ are symmetric monoidal categories with strict monoidal underlying functors into $\mathbb{C}$. The same holds for the categories ${}_c\mathbf{Mon}\mathbb{C}$ of commutative monoids in $\mathbb{C}$ and ${}_{coc}\mathbf{Comon}\mathbb{C}$ of cocommutative comonoids in $\mathbb{C}$. Consequently, the categories $\mathbf{Bialg}_R$, ${}_c\mathbf{Bialg}_R$ and ${}_{coc}\mathbf{Bialg}_R$ are monoidal. $\mathbf{Hopf}_R$, ${}_c\mathbf{Hopf}_R$ and ${}_{coc}\mathbf{Hopf}_R$ are also monoidal.*

Fact 2 (Eckmann-Hilton Principle)**.** *For every symmetric monoidal category $\mathbb{C}$, the category $\mathbf{Mon}(\mathbf{Mon}\mathbb{C})$ of monoids in $\mathbf{Mon}\mathbb{C}$, the category of monoids in $\mathbb{C}$, coincides with ${}_c\mathbf{Mon}\mathbb{C}$, the category of commutative monoids in $\mathbb{C}$.*

Dually: $\mathbf{Comon}(\mathbf{Comon}\mathbb{C}) = {}_{coc}\mathbf{Comon}\mathbb{C}$.

Fact 3. *Every strict monoidal functor $F\colon \mathbb{C} \to \mathbb{C}'$ induces functors $\hat{F}\colon \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$ and (dually) $F^*\colon \mathbf{Comon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}'$ such that*

$$\begin{array}{ccc} \mathbf{Mon}\mathbb{C} & \xrightarrow{\hat{F}} & \mathbf{Mon}\mathbb{C}' \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C}' \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{Comon}\mathbb{C} & \xrightarrow{F^*} & \mathbf{Comon}\mathbb{C}' \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C}' \end{array}$$

If F has a right adjoint, then $\hat{F}$ has a right adjoint and, dually:

If F has a right adjoint, then F^* has a right adjoint.

These adjoints commute with the underlying functors $|-|$ as well.

Fact 4. *For every symmetric monoidal category $\mathbb{C}$ the following hold.*

1. *${}_c\mathbf{Mon}\mathbb{C}$ is closed in $\mathbf{Mon}\mathbb{C}$ under limits.*
2. *${}_{coc}\mathbf{Comon}\mathbb{C}$ is closed in $\mathbf{Comon}\mathbb{C}$ under colimits.*

If, for every object C in $\mathbf{C}$, the functor $C \mapsto C \otimes -$ preserves (for any regular cardinal λ) λ-directed colimits then ${}_c\mathbf{Mon}\mathbb{C}$ is closed in $\mathbf{Mon}\mathbb{C}$ under λ-directed colimits as well.

3. Limits and Colimits in $\mathbf{Hopf}_R$

As shown in [5], colimits in $\mathbf{Hopf}_R$ are formed as in $\mathbf{Bialg}_R$ and the latter are the respective colimits in $\mathbf{Alg}_R$ supplied with the unique coalgebra structure making the colimiting maps in $\mathbf{Alg}_R$ also coalgebra homomorphism. In other words, if—for example—$(\mathsf{H_i}, S_i)_{i\in I}$ is a family of Hopf algebras and $\lambda_i\colon H_i^a \to H$ a coproduct in $\mathbf{Alg}_R$ of its underlying algebras H_i^a, then a coproduct of the family $(H_i, S_i)_{i\in I}$ in $\mathbf{Hopf}_R$ is $(\mathsf{H}, S) = (H, \Delta, \epsilon, S)$ where $\Delta\colon \mathsf{H} \to \mathsf{H}\otimes\mathsf{H}$ and $\epsilon\colon \mathsf{H} \to R$ are the unique coalgebra homomorphisms, such that the diagrams

$$\begin{array}{ccc} H_i & \xrightarrow{\lambda_i} & H \\ {\scriptstyle\Delta_i}\downarrow & & \downarrow{\scriptstyle\Delta} \\ H_i\otimes H_i & \xrightarrow[\lambda_i\otimes\lambda_i]{} & H\otimes H \end{array} \qquad\qquad \begin{array}{ccc} H_i & \xrightarrow{\lambda_i} & H \\ & {\scriptstyle\epsilon_i}\searrow & \downarrow{\scriptstyle\epsilon} \\ & & R \end{array} \tag{1}$$

commute, while $S\colon \mathsf{H} \to \mathsf{H}^{\mathsf{op,cop}}$ is the unique bialgebra homomorphism making the diagram

$$\begin{array}{ccc} H & \overset{S}{\dashrightarrow} & H^{\mathsf{op,cop}} \\ {\scriptstyle\lambda_i}\uparrow & & \uparrow{\scriptstyle\lambda_i} \\ H_i & \xrightarrow[S_i]{} & H_i^{\mathsf{op,cop}} \end{array} \tag{2}$$

commute. Note that, for $R = k$ a field, the case of coproducts indeed is already contained in [8].

Limits can be constructed dually, provided that R is von Neumann regular.

This is enough to prove the following results concerning closure under limits and colimits respectively of the various subcategories we are interested in.

Proposition 1. *In the chains of subcategories*

$$_{coc}\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R \subset \mathbf{Bialg}_R$$

$$_{coc}\mathbf{Hopf}_R \subset {}_{coc}\mathbf{Bialg}_R \subset \mathbf{Bialg}_R$$

each subcategory is closed under all colimits in each of its successors. In particular, each of these categories is cocomplete.

Proof. To show that $_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R$ is closed under colimits one has, by the above, only to prove that the morphism S in diagram (2) is bijective if all the S_i are bijective. Since bijective bialgebra homomorphisms are isomorphisms of bialgebras, one has the following commutative diagram in $\mathbf{Bialg}_R$

$$\begin{array}{ccccc} H^{\mathsf{op,cop}} & \overset{T}{\dashrightarrow} & H & \xrightarrow{S} & H^{\mathsf{op,cop}} \\ \uparrow{\scriptstyle\lambda_i} & & {\scriptstyle\lambda_i}\uparrow & & \uparrow{\scriptstyle\lambda_i} \\ H_i^{\mathsf{op,cop}} & \xrightarrow[S_i^{-1}]{} & H_i & \xrightarrow[S_i]{} & H_i^{\mathsf{op,cop}} \end{array}$$

where T is the bialgebra homomorphism induced by the family $(S_i^{-1})_i$ (note that the left hand column is a colimit as well). The equations $S_i \circ S_i^{-1} = \mathsf{id}$ for each i then induce $S \circ T = \mathsf{id}$ by the colimit property. $T \circ S = \mathsf{id}$ is proved analogously.

A similar argument shows that $_{S^2=\mathsf{id}}\mathbf{Hopf}_R$ is closed under colimits in $\mathbf{Hopf}_R$ and, thus, in $_{bi}\mathbf{Hopf}_R$.

Since $_{coc}\mathbf{Bialg}_R = {}_{coc}\mathbf{Comon}(\mathbf{Alg}_R) \subset \mathbf{Comon}(\mathbf{Alg}_R) = \mathbf{Bialg}_R$, closure of colimits here is a special instance of Fact 4 (2).

Now closure of $_{coc}\mathbf{Hopf}_R$ under colimits in $\mathbf{Hopf}_R$ as well as in $\mathbf{Bialg}_R$ is obvious, since colimits in $\mathbf{Hopf}_R$ are formed on the level of $\mathbf{Bialg}_R$.

Cocompleteness of the categories under consideration now follows from cocompleteness of $\mathbf{Bialg}_R$ and $\mathbf{Hopf}_R$ respectively. □

Proposition 2. *If R is von Neumann regular, then in the chains of subcategories*

$$_{c}\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R \subset \mathbf{Bialg}_R$$

$$_{c}\mathbf{Hopf}_R \subset {}_{c}\mathbf{Bialg}_R \subset \mathbf{Bialg}_R$$

each subcategory is closed under all limits (and λ-directed colimits for each λ) in each of its successors. ${}_c\mathbf{Bialg}_R$ *is closed under all limits in* $\mathbf{Bialg}_R$ *even for arbitrary rings.*

Proof. The proofs are dual (for directed colimits analogous; clearly in $\mathbf{Mod}_R$ each functor $C \mapsto C \otimes -$ preserves (directed) colimits) to the arguments in the previous proof. □

4. The Adjunctions

4.1. Reflectivity and Coreflectivity.

Proposition 3. *For any ring R, in the chains of subcategories*

$$ {}_{coc}\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R $$

$$ {}_{coc}\mathbf{Hopf}_R \subset {}_{coc}\mathbf{Bialg}_R \subset \mathbf{Bialg}_R $$

every category is locally presentable and coreflective in each of its successors.

Proof. All categories can be represented as equifiers of families of pairs of natural transformations between accessible functors (analogous to the respective arguments in [3] and [4]). Thus, they are accessible categories by [1, 2.76]. Being cocomplete by Proposition 1 they then are locally presentable by [1, 2.47]. Coreflectivity then follows by the Special Adjoint Functor Theorem. □

Proposition 4. *For any von Neumann regular ring R, in the chains of subcategories*

$$ {}_c\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R $$

$$ {}_c\mathbf{Hopf}_R \subset {}_c\mathbf{Bialg}_R \subset \mathbf{Bialg}_R $$

every category locally presentable and reflective in each of its successors.
${}_c\mathbf{Bialg}_R$ *is locally presentable and reflective in* $\mathbf{Bialg}_R$ *even for every commutative ring R.*

Proof. All categories are, by Proposition 2, closed under limits and λ-directed colimits for any λ. Since $\mathbf{Hopf}_R$ and $\mathbf{Bialg}_R$ are locally presentable (see [5]), all categories then are locally presentable as well and reflective in their respective supercategories by [1, 2.48]. □

Corollary 1. *For any von Neumann regular ring R, in the chain of subcategories*

$$ {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R $$

every category is coreflective in each of its successors.

Proof. By the previous results all embeddings preserve colimits and their domains are locally presentable. They, thus, have right adjoints again by the Special Adjoint Functor Theorem. □

Corollary 2.
(1) *For every commutative ring R,* ${}_{c,coc}\mathbf{Bialg}_R$ *is reflective in* ${}_{coc}\mathbf{Bialg}_R$ *and coreflective in* ${}_c\mathbf{Bialg}_R$.
(2) ${}_{c,coc}\mathbf{Hopf}_R$ *is reflective in* ${}_{coc}\mathbf{Hopf}_R$ *for every von Neumann regular ring R and coreflective in* ${}_c\mathbf{Hopf}_R$ *for every commutative ring R.*
(3) ${}_c\mathbf{Hopf}_R$ *is coreflective in* ${}_c\mathbf{Bialg}_R$ *for every commutative ring R.*
(4) ${}_{coc}\mathbf{Hopf}_R$ *is reflective in* ${}_{coc}\mathbf{Bialg}_R$ *for every von Neumann regular ring R.*
(5) ${}_{c,coc}\mathbf{Hopf}_R$ *is reflective and coreflective in* ${}_{c,coc}\mathbf{Bialg}_R$ *for every von Neumann regular ring R.*

Proof. Observe that, by the Eckmann-Hilton Principle, one has

$$ {}_c\mathbf{Bialg}_R = \mathbf{Mon}(\mathbf{Bialg}_R) \tag{3} $$

$$ {}_{coc}\mathbf{Bialg}_R = \mathbf{Comon}(\mathbf{Bialg}_R) \tag{4} $$

$$ {}_{c,coc}\mathbf{Bialg}_R = \mathbf{Mon}({}_{coc}\mathbf{Bialg}_R) \tag{5} $$

$$ {}_{c,coc}\mathbf{Bialg}_R = \mathbf{Comon}({}_c\mathbf{Bialg}_R) \tag{6} $$

$$ {}_c\mathbf{Hopf}_R = \mathbf{Mon}(\mathbf{Hopf}_R) \tag{7} $$

$$ {}_{coc}\mathbf{Hopf}_R = \mathbf{Comon}(\mathbf{Hopf}_R) \tag{8} $$

$$ {}_{c,coc}\mathbf{Hopf}_R = \mathbf{Mon}({}_{coc}\mathbf{Hopf}_R) \tag{9} $$

$$ {}_{c,coc}\mathbf{Hopf}_R = \mathbf{Comon}({}_c\mathbf{Hopf}_R) \tag{10} $$

Now reflectivity of ${}_{c,coc}\mathbf{Bialg}_R$ in ${}_{coc}\mathbf{Bialg}_R$, for example, follows from Fact 3 by using equations (4) and (6): The embedding $E\colon {}_c\mathbf{Bialg}_R \hookrightarrow \mathbf{Bialg}_R$ has a left adjoint by Proposition 4 and it is a trivial observation that the embedding ${}_{c,coc}\mathbf{Bialg}_R \hookrightarrow {}_{coc}\mathbf{Bialg}_R$ is, in the notation of Fact 3, nothing but E^*.

All other claims follow analogously (for statements 3 and 4 use, instead of Propositions 4 and 3 the (co)reflectivity results of [5]). □

Remark 1. Some of the results above can, in the restricted case of a von Neumann regular ring, also be obtained by using the explicit construction of the Hopf (co)reflection as presented in [6]. Recall that, given a bialgebra B, one constructs its (co) reflection as follows: Define a family of bialgebras $(B_n)_{n\in\mathbb{N}}$ by $B_0 := B$ and $B_{n+1} := B_n^{\mathsf{op},\mathsf{cop}}$. Then the Hopf reflection RB of B is a (suitable) homomorphic image of $\coprod B_n$ while the Hopf coreflection of B is a (suitable) subbialgebra of $\prod B_n$.

Now, obviously, if B is commutative (cocommutative, commutative and cocommutative) so is each B_n and then $\coprod B_n$ and $\prod B_n$ respectively (since the functors on $\mathbf{Bialg}_R$ sending B to B^{op} or B^{cop} are isomorphisms and therefore preserve (co)products). It is, moreover, easy to see that images and subbialgebras of a commutative (cocommutative, commutative and cocommutative) bialgebra have the respective property again. Thus the Hopf (co)reflection of a commutative (cocommutative, commutative and cocommutative) bialgebra is a commutative (cocommutative, commutative and cocommutative) Hopf algebra.

4.2. **Monadicity.** Extending the results from [5] that $\mathbf{Hopf}_R$ is comonadic over $\mathbf{Alg}_R$ (always) and monadic over $\mathbf{Coalg}_R$, provided that R is von Neumann regular, we also get

Proposition 5. *For every von Neumann regular ring R, the following hold:*

(1) (a) *The cofree Hopf algebra on a commutative algebra A is commutative and, thus, the cofree commutative Hopf algebra on A.*
   (b) *${}_c\mathbf{Hopf}_R$ is comonadic over $\mathbf{Alg}_R$.*
(2) (a) *The free Hopf algebra on a cocommutative coalgebra C is cocommutative and, thus, the free cocommutative Hopf algebra on C.*
   (b) *${}_{coc}\mathbf{Hopf}_R$ is monadic over $\mathbf{Coalg}_R$.*

Statement 1 (a) even holds for an arbitrary commutative ring R.

Proof. As in the arguments for the previous Corollary use equation (7) and the fact that $\mathbf{Mon}(\mathbf{Alg}_R) = {}_c\mathbf{Alg}_R$. The lift $\hat{V}$ of forgetful functor $V\colon \mathbf{Hopf}_R \to \mathbf{Alg}_R$ then is the forgetful functor ${}_c\mathbf{Hopf}_R \to {}_c\mathbf{Alg}_R$. By Fact 3 this functor has a right adjoint $\tilde{G}$, since V has a right adjoint G (see [5]) and the digram

$$\begin{array}{ccc} {}_c\mathbf{Alg}_R & \xrightarrow{\tilde{G}} & {}_c\mathbf{Hopf}_R \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathbf{Alg}_R & \xrightarrow[G]{} & \mathbf{Hopf}_R \end{array}$$

commutes. But this commutativity precisely says, that the cofree Hopf algebra on a commutative algebra A is commutative. This proves 1 (a). Comonadicity now follows by Beck's Theorem (see e.g. [2]) since $\mathbf{Hopf}_R \to \mathbf{Alg}_R$ is monadic and ${}_c\mathbf{Hopf}_R$ is closed in $\mathbf{Hopf}_R$ under limits.

Statement 2 follows dually. □

The following now is easy to prove, too, where the second statement generalizes a result of [8].

Proposition 6. (1) *Let R be an arbitrary commutative ring. Then there exists a cofree cocommutative R-Hopf algebra over any R-algebra.*
(2) *Let R be a von Neumann regular ring. Then there exists a free commutative R-Hopf algebra over any R-coalgebra.*

Proof. Over every algebra A there exists a cofree Hopf algebra H_A by [5]. By composition of adjunctions (see e.g. [2]) the coreflection of H_A into ${}_{coc}\mathbf{Hopf}_R$, which exists by Proposition 3, is cofree over A.

The second statement follows analogously. □

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