Residually Small Varieties Without Rank

Hans–E. Porst

Dedicated to Nico Pumplün on the occasion of his 70th birthday

Abstract

Subdirect representations are investigated in varieties which are defined by operations of not necessarily finite arity. It is shown that, in this context, Birkhoff's Subdirect Representation Theorem does not hold. However, a class of unranked varieties is identified which admit subdirect representations by subdirectly irreducibles and then even are residually small.

Introduction

The famous Birkhoff Subdirect Representation Theorem states that, given any algebra A in some finitary variety $\mathcal{V}$, there exists, in $\mathcal{V}$, a subdirect representation of A by subdirectly irreducible algebras A_i. In detail: given A, there exists a family $m_i\colon A \to A_i,\ i \in I$, of homomorphisms such that

(i) the family $(m_i)_{i\in I}$ is point-separating,

(ii) each m_i is surjective,

and

(iii) each A_i has the property that any point-separating family of surjective homomorphisms $n_j\colon A_i \to B_j\ (j \in J)$ contains an isomorphism.

If the subdirectly irreducible algebras in $\mathcal{V}$, i.e., those satisfying (iii) above, form — up to isomorphism — a set (and not a proper class), $\mathcal{V}$ is called *residually small*. This is a very restrictive property on $\mathcal{V}$ satisfied, e.g., by the varieties of Abelian groups and Boolean algebras, but not by the varieties of groups or rings.

Thus, the Birkhoff Theorem tells us that there are always "enough" subdirectly irreducible algebras in a given (finitary) variety $\mathcal{V}$, while residual smallness of $\mathcal{V}$ means that there are not "too many".

These questions certainly can also be asked for non–finitary varieties, in particular for unranked ones (i.e., for varieties of algebras which need operations of arbitrarily large arities) as, e.g., the category CaBa of complete atomic Boolean algebras.

We will show by some simple examples, that these varieties sometimes do but in general do not have enough subdirectly irreducible algebras. This will be explained partially by an analysis of the standard proof of the subdirect Representation Theorem, which shows that this proof heavily depends on the finiteness of arities. It is then all the more surprising that we can construct a whole class of unranked varieties for which the Birkhoff Subdirect Representation Theorem holds, and which, in addition, are residually small. This then is an extension of Birkhoff's classical result quite different in nature from the extension to "finitary generalized varieties" as presented in [7].

1 Prerequisites and notations

In the sequel subdirect representations will often be obtained by dualization. We therefore recall the categorized versions of the respective notions and their duals as follows.

Definition 1 ([7]) A family of morphisms $m_i\colon A \to A_i,\ i \in I$, in a category A is called a *subdirect representation* of A provided that

(i) the family $(m_i)_{i\in I}$ is jointly monomorphic,

and

(ii) each m_i is a regular epimorphism.

An A–object A is called *subdirectly irreducible* if every subdirect representation of A contains an isomorphism.

Note that, in case A is a variety these notions coincide with the classical ones. The duals of these concepts have been called "conjunct representation" and "conjunctly irreducible" respectively by H.P. Gumm and T. Schröder [6], and we maintain these notions. That is, a jointly epimorphic family of regular monomorphisms $e_i\colon A_i \to A,\ i \in I$, is a *conjunct representation* of A, and A is *conjunctly irreducible* provided that any conjunct representation of A contains an isomorphism.

We recall from categorical algebra the following facts:

Proposition 1 (see e.g. **[5, 2]**) *1. If $U\colon \mathsf{A} \to \mathsf{Set}$ is a monadic functor into the category Set of sets, then A is concretely equivalent to a variety (with possibly a class of operations, in particular, not necessarily with a bound on the arities of the defining operations).*

2. For every monadic functor $V\colon \mathsf{A} \to \mathsf{Set}^{\mathsf{op}}$ the composition of V with the contravariant powerset functor $\mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$ is monadic.

Recall also that a variety defined by operations whose arities are all less than a given regular cardinal λ is a locally λ-presentable category and that its underlying functor is λ-*accessible*, i.e., it preserves λ-directed colimits.

From coalgebra we recall the following: Given a set functor F one considers the category $\mathsf{Coalg}F$ of F*–coalgebras* having as objects pairs $(X, X \xrightarrow{\alpha_X} FX)$, with X a set and α_X a map, and having as morphisms $f\colon (X, \alpha_X) \to (Y, \alpha_Y)$ the *coalgebra homomorphisms*, i.e., those maps $f\colon X \to Y$ for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & FX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ Y & \xrightarrow[\alpha_Y]{} & FY \end{array}$$

commutes. By $U\colon \mathsf{Coalg}F \to \mathsf{Set}$ we denote the obvious underlying functor.

Then the following hold (see e.g. [2]).

Proposition 2 *For every accessible set functor F the following hold:*

1. *The category $\mathsf{Coalg}F$ is a locally presentable category.*
2. *The functor $U\colon \mathsf{Coalg}F \to \mathsf{Set}$ is comonadic.*

Thus, given an accessible set functor F, the category $(\mathsf{Coalg}F)^{\mathsf{op}}$ is monadic over Set and hence equivalent to a variety.

Definition 2 The variety equivalent to the category $(\mathsf{Coalg}F)^{\mathsf{op}}$ for an accessible set functor F is called the *variety determined by F* and will be denoted by $\mathcal{V}_F$.

Note that $\mathcal{V}_F$ is unranked, for otherwise it would be locally presentable; and this would contradict statement 1. of the above proposition, since categories together with their duals can be locally presentable in trivial cases only ([4]). The simplest example is the variety CaBa of complete atomic Boolean algebras: CaBa is just $\mathcal{V}_F$ for the constant functor F mapping each set to a singleton.

We will need the following coalgebraic lemma:

Lemma 1 *Let F preserve intersections. Then for any family (C_i, α_i), $i \in I$, of subcoalgebras*[1] *of an F–coalgebra (C, α_C), the intersection $\cap_I C_i$ carries the structure of a subcoalgebra of (C, α_C). In particular, for every $c \in C$ there exists a smallest subcoalgebra $\langle c \rangle$ of (C, α_C) containing c.*

2 A closer look at the unranked case

Complete atomic Boolean algebras

Since the category CaBa of complete atomic Boolean algebras is dually equivalent to the category Set, we start by investigating conjunct representations in Set.

Lemma 2 *Every set X has a conjunct representation by singleton sets.*

Proof Denote, for $x \in X$, by $\iota_x\colon \{x\} \to X$ the embedding. The family $(\iota_x)_{x \in X}$ clearly is a conjunct representation. ◇

[1] (D, α_D) is called *subcoalgebra* of (C, α_C), provided that $D \subset C$ and the embedding $i\colon D \to C$ is a coalgebra homomorphism.

Lemma 3 *The conjunctly irreducible sets are precisely the singleton sets.*

Proof The previous lemma shows that no set with more than one element is conjunctly irreducible. And the empty set fails to be conjunctly irreducible: consider the respective (empty) family of the proof above. ◇

By dualization one thus obtains

Corollary 1 *Every algebra in* CaBa *has a subdirect representation by subdirectly irreducible algebras and, in addition,* CaBa *is residually small.*

We will see later, that this is a special instance of a far more general result.

Commutative unital C^*–algebras

The category $\mathsf{cC^*Alg_1}$ of commutative unital C^*–algebras is known to be a variety of algebras given by a finite set of finitary operations and one operation of countable arity (see [5]). Since $\mathsf{cC^*Alg_1}$ is dually equivalent to the category of compact Hausdorff spaces we investigate conjunct representations of the latter. In fact, literally the same arguments as for Set show:

Lemma 4 *Every compact Hausdorff space has a conjunct representation by spaces consisting of one element, and these spaces are precisely the conjunctly irreducible ones.*

Again by dualization we obtain

Corollary 2 *Every commutative unital C^*–algebra has a subdirect representation by subdirectly irreducibles and, moreover, the category* $\mathsf{cC^*Alg_1}$ *is residually small.*

Compact Hausdorff Spaces

The category $\mathsf{Comp_2}$ of compact Hausdorff spaces is well known to be monadic, in fact an unranked variety.

Lemma 5 *Every compact Hausdorff space has a subdirect representation by the set of closed subspaces of the unit interval I.*

Proof Let X be compact Hausdorff and $x, y \in X,\ x \neq y$. By Urysohn's Lemma there exists a continuous map $\varphi_{x,y}\colon X \to I$ with $\varphi_{x,y}(x) = 0$ and $\varphi_{x,y}(y) = 1$. Denote the corestriction of $\varphi_{x,y}$ to its image by $\eta_{x,y}\colon X \to K_{x,y}$. Now the family $\eta_{x,y}\colon X \to K_{x,y},\ x, y \in X,\ x \neq y$, is the desired representation. ◇

Lemma 6 *The subdirectly irreducible spaces are precisely the discrete spaces with at most two elements.*

Proof It is easy to see that these spaces are subdirectly irreducible. If X has at least three points construct, again by Urysohn's Lemma, for any three different points $x, y, z \in X$ a map $\varphi = \varphi_{\{x,y\},z}\colon X \to I$ with $\varphi(x) = \varphi(y) = 0$ and

$\varphi(z) = 1$. This is a point–separating family, no member of which is injective. Factorizing this family over the images of its members produces a subdirect representation with no injective, hence no bijective, member.[2] $\diamond$

It then follows that any compact Hausdorff space admitting a subdirect representation by subdirectly irreducibles is a subspace of a power of the two-element discrete space and thus zero-dimensional. We therefore have the following

Corollary 3 *Not every compact Hausdorff space has subdirect representation by subdirectly irreducibles.*

The last example shows that, despite the positive results for CaBa and $\mathsf{cC^*Alg}_1$, one cannot expect Birkhoff's subdirect Representation Theorem to hold in non–finitary varieties. In fact one should not expect so as a closer look at its standard proof (see e.g. [1]) shows: it depends on a Zorn's Lemma argument which requires unions of chains of congruences to be congruences again: and to establish this one needs finiteness of the arities of all defining operations.

3 A class of residually small varities

In this section we will construct a class of unranked varieties in which Birkhoff's Subdirect Representation Theorem holds and which, in addition, are residually small. Our first example, CaBa, is the simplest case of this construction. In the sequel F always denotes a set functor.

Using Lemma 1 we can prove (see also [6]):

Proposition 3 *Let F preserve intersections. The conjunctly irreducible F–coalgebras are precisely the coalgebras of the form $\langle c\rangle$ with $c \in C$ for some F–coalgebra (C, α_C), and every coalgebra has a conjunct representation of those.*

Proof Let $f_i\colon (C_i, \alpha_i) \to \langle c\rangle$, $i \in I$ be a conjunct representation. Chose $k \in I$ such that $f_k(c_k) = c$ for some $c_k \in C_k$; then the image of f_k, being a subcoalgebra of $\langle c\rangle$, must be all of $\langle c\rangle$. Thus f_k is an isomorphism.

Conversely, let (C, α_C) be conjunctly irreducible. Since the family of embeddings $i_c\colon \langle c\rangle \to (C, \alpha_C)$ clearly is a conjunct representation we conclude $\langle c_0\rangle = (C, \alpha_C)$ for some $c_0 \in C$. $\diamond$

Again by dualization we obtain

Theorem 1 *Let F preserve intersections. Then every algebra in $\mathcal{V}_F$ has subdirect representations by subdirectly irreducibles.*

From the theory of coalgebras we finally recall the following result (see e.g. [6] or [3]).

[2] This simple argument is due to Christoph Schubert.

Proposition 4 *Let F be λ–accessible. Then for every F–coalgebra (C, α_C) and any $c \in C$ there exists a subcoalgebra (D, α_D) of (C, α_C) with $c \in D$ and $\mathsf{card} D \leq \lambda$.*

It follows from the above that $\mathsf{Coalg} F$ contains, up to isomorphism, only a set of coalgebras of the form $\langle x \rangle$, provided F is accessible. Combining this with the previous Theorem we get

Theorem 2 *The Birkhoff Subdirect Representation Theorem holds in the class of all varieties $\mathcal{V}_F$, where F is an accessible set functor which, in addition, preserves intersections. Moreover, each variety $\mathcal{V}_F$ in this class is residually small.*

Examples of functors satisfying the hypotheses of this theorem are the so called (generalized) *polynomial functors* $X \mapsto \sum_{n<\lambda} A_n \times X^n$. Since CaBa is $\mathcal{V}_F$ for the polynomial functor $X \mapsto 1$, our first example is a special instance of Theorems 1 and 2.

References

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Department of Mathematics, University of Bremen, 28359 Bremen, Germany
porst@math.uni-bremen.de