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On Varieties and Covarieties in a Category

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A concept of equation morphism is introduced for every endofuctor F of a cocomplete category C. Equationally defined classes of F–algebras for which free algebras exist are called varieties. Every variety is proved to be monadic over C, and conversely, every monadic category is equivalent to a variety. And the Birkhoff Variety Theorem is proved for "Set–like" categories.
By dualizing, we arrive at a concept of coequation such that covarieties, i.e., coequationally specified classes of coalgebras with cofree objects, precisely correspond to comonadic categories. Natural examples of covarieties are presented.

Introduction

Our paper whose extended abstract has appeared in (Adámek and Porst 2001) is devoted to a concept of variety of algebras in an arbitrary category C. The basic examples for $\mathsf{C} = \mathsf{Set}$ include

(1) all finitary varieties of classical universal algebra (groups, lattices, etc.),
(2) all infinitary varieties of bounded arities (e.g., Boolean σ–algebras, σ–complete lattices), and
(3) those infinitary varieties of unbounded arities that have free algebras, e.g., complete semilattices and compact Hausdorff spaces.

It is well–known that finitary varieties precisely correspond to finitary monads over Set (see (Lawvere 1963) or (Manes 1976)) — and it is a folklore that this generalizes to (some concept of) infinitary varieties and arbitrary monads over Set. The result that we consider surprising is that such a correspondence can be formulated naturally over every cocomplete category C. Recall from (Adámek 1974) that for every endofunctor F preserving colimits of ω–chains a free F–algebra $X^\sharp$ on an arbitrary object X can be

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described as an ω–colimit of the following ω–chain (that mimics the iterative construction of the set of all terms in variables from X):

$$0 \to F0 + X \to F(F0 + X) + X \to \cdots$$

If F does not preserve ω–colimits, we might iterate further in order to obtain a free algebra $X^\sharp$: on limit steps we form a colimit of the previous chain, and isolated steps take W_i and answer with $W_{i+1} = FW_i + X$ ($i \in \mathsf{Ord}$). If the connecting morphism $W_i \to W_{i+1}$ is an isomorphism for some i, then $W_i = X^\sharp$ is a free F–algebra. Here, however, we do not restrict ourselves to endofunctors having free algebras: we simply work instead with the whole transfinite construction $(W_i)_{i\in\mathsf{Ord}}$. It has the following property: for every F–algebra A and every morphism $f\colon X \to A$ (interpretation of variables) we obtain a unique cocone $f_i^\sharp\colon W_i \to A$ of the above chain such that the second component of $f_{i+1}^\sharp\colon FW_i + X \to A$ is f (for all ordinals i). We can think of $f_i^\sharp$ as computation of terms of depth i.

This enables us to define an *equation* as a parallel pair of morphisms whose codomain is one of these W_i's:

$$U \underset{v}{\overset{u}{\rightrightarrows}} W_i \quad (i \in \mathsf{Ord})$$

An algebra A is said to *satisfy* that equation provided that for every morphism $f\colon X \to A$ we have $f_i^\sharp u = f_i^\sharp v$. And a full subcategory $\mathsf{V} \hookrightarrow \mathsf{Alg}(F)$ of the category of all F–algebras is called a *variety* if

i. V is specified by a class of equations, and
ii. V–free algebras exist, i.e., the natural forgetful functor $\mathsf{V} \to \mathsf{C}$ has a left adjoint.

This includes e.g. complete semilattices over Set (since free complete semilattices are the power–sets) but not complete lattices (where a free object on 3 generators does not exist—see (Hales 1964)). For any cocomplete category C, varieties are proved to be precisely the monadic categories over C.

The above concept of equation has been, for $\mathsf{C} = \mathsf{Set}$, introduced in (Adámek 1983). A closely related concept, for C arbitrary, has been studied by Jan Reiterman in his dissertation, see also (Reiterman 1978), where quotient chains of the above chain (W_i) are considered as equations.

Functors F such that every object generates a free F–algebra are called *varietors* in (Adámek and Trnková 1990). Although varieties are defined, in general, without this assumption, one can say more about algebras over varietors. E.g., one can prove a generalization of the Birkhoff Variety Theorem for all varietors on Set (see (Adámek 1983, Thm 5D5)). In the present paper we prove such a theorem for every complete, regularly cowellpowered category with regular factorizations; a similar result can be found in the above mentioned paper of Jan Reiterman. We can also work with coequalizers $c\colon W_i \to C$ of the above pairs u, v rather than the pairs per se. Then to satisfy $u = v$ is the same as to be injective w.r.t. c. A generalization of varieties using the concept of injectivity has been studied in (Herrlich and Ringel 1972) and (Banaschewski and Herrlich 1976). The generalization of Birkhoff's Variety Theorem requests a study of limits and colimits that we undertake, too.

All these considerations dualize immediately to *coalgebras* over an endofunctor F of C, since the category $\mathsf{Coalg}(F)$ of coalgebras is dual to $\mathsf{Alg}(F^{op})$. Thus, we obtain a general concept of coequation and coequationally defined classes of coalgebras. In case of a covarietor F on Set, a coequation is simply an element of a cofree coalgebra, which is the concept studied in (Gumm 1999). In the present paper we show that this is just a special case of a very general concept: whenever C is a complete category, then comonadic categories over C are precisely the covarieties. And for $\mathsf{C} = \mathsf{Set}$ and for covarietors F (which includes, e.g., all bounded functors) the Birkhoff Covariety Theorem is proved to hold — thus our concept of covariety is identical with that of (Rutten 2000) and with other concepts in the literature. For example, with covarieties defined by means of projectivity (i.e., a dualization of the above mentioned results of Banaschewski and Herrlich) in (Gumm and Schröder 1998), (Kurz 2000) and (Awoday and Hughes 2000). Covarieties satisfying additional assumptions (i.e., a concept not equivalent to that studied in the present paper) have been studied e.g. in (Cîrstea 2000) and (Ghani et al. 2001).

1. Algebras and Coalgebras

1.1 Standing assumption Throughout the paper C denotes a category and F an endofunctor of C. Our main interest will be in $\mathsf{C} = \mathsf{Set}$ or Set^{op}, but in general nothing is assumed about C. The dual of a functor $G\colon \mathsf{C} \to \mathsf{D}$, i.e., the functor $\mathsf{C}^{op} \to \mathsf{D}^{op}$ acting on objects and morphisms as G, will be denoted by G^{op}.

1.2 Category $\mathsf{Alg}(F)$ of F–algebras: Objects of $\mathsf{Alg}(F)$, called F–algebras, are pairs (C, α_C) where C is an object of C and $\alpha_C\colon FC \to C$ is a morphism. Morphisms $f\colon (C, \alpha_C) \to (D, \alpha_D)$ of $\mathsf{Alg}(F)$, called *F–algebra homomorphisms*, are morphisms $f\colon C \to D$ of C such that $f \circ \alpha_C = \alpha_D \circ Ff$. Composition and identities in $\mathsf{Alg}(F)$ are those of C. We denote by

$$U_F\colon \mathsf{Alg}(F) \to \mathsf{C}^{\dagger}$$

the canonical forgetful functor $(C, \alpha_C) \mapsto \mathsf{C}$.

1.3 Category $\mathsf{Coalg}(F)$ of F–algebras: Objects of $\mathsf{Coalg}(F)$, called *F–coalgebras*, are pairs (C, α_C) where C is an object of C and $\alpha_C\colon C \to FC$ is a morphism. Morphisms $f\colon (C, \alpha_C) \to (D, \alpha_D)$ of $\mathsf{Coalg}(F)$, called *F–coalgebra homomorphisms*, are morphisms $f\colon C \to D$ of C such that $Ff \circ \alpha_C = \alpha_D \circ f$. Composition and identities are again those of C. We denote by

$${}_FU\colon \mathsf{Coalg}(F) \to \mathsf{C}^{\dagger}$$

the canonical forgetful functor $(C, \alpha_C) \mapsto C$.

1.4 Lemma *For any functor $F\colon \mathsf{C} \to \mathsf{C}$ the following hold:*

1) $\mathsf{Coalg}(F) = (\mathsf{Alg}(F^{op})^{op})$, *i.e., F–coalgebras from the dual of the category of F^{op}–algebras;*
2) ${}_FU = (U_{F^{op}})^{op}$.

† Whenever confusion is unlikely to arise we will omit the subscript F.

1.5 Examples (1) Algebras on one binary and one nullary operation are F–algebras where F is the endofunctor of Set given by $FX = (X \times X) + 1$.

F–coalgebras, i.e., sets C equipped with a function $\alpha_C : C \to C \times C + 1$, can be viewed as dynamic systems with two inputs and deadlocks. That is, C is the set of all states, and for every state $c \in C$ either $\alpha_C(c)$ is the pair of states which are the reactions of c to the two inputs, respectively, or c is a deadlock state. Morphisms are the functions preserving reaction to inputs and preserving and reflecting deadlock.

(2) Finitary universal algebras of a given (one–sorted) signature $\Omega = (\Omega_n)_{n\in\mathbb{N}}$, where Ω_n is the set of all n–ary operation symbols, can be viewed as F–algebras for the following endofunctor $F = F_\Omega$ of Set: F_Ω assigns to a set X the set $\sum_{n\in\mathbb{N}} \Omega_n \times X^n$. Correspondingly F_Ω assigns to a map $f : X \to Y$ the map $\sum_{n\in\mathbb{N}} \Omega_n \times f^n$, i.e., the map $\sum_{n\in\mathbb{N}} \Omega_n \times X^n \to \sum_{n\in\mathbb{N}} \Omega_n \times Y^n$ mapping a pair $(\omega, (x_1, \dots, x_n))$ to the pair $(\omega, (fx_1, \dots, fx_n))$.

F_Ω–coalgebras can be viewed as dynamical systems with several inputs and side effects. This has been demonstrated in (Rutten 2000).

(3) Many–sorted algebras (of a given S–sorted signature Ω) can, analogously, be described as F_Ω–algebras for the corresponding endofunctors F_Ω of Set^S. For example, the category of multigraphs and homomorphisms is the category of 2-sorted algebras of sorts v (vertices) and e (edges) with two unary operations *target, source*: $e \to v$. This is also a category of coalgebras (for the endofunctor F on $\mathsf{Set} \times \mathsf{Set}$ given by $F(X_v, X_e) = (1, X_v \times X_v)$)—in fact, every category of unary algebras is a category of coalgebras as well.

(4) Denote by

$$\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$$

the power–set functor. $\mathcal{P}$–algebras are sets A endowed with an operation $\alpha : \mathcal{P}A \to A$. For example, a complete join semilattice is described by $\alpha : M \mapsto \vee M$ (see Example 3.5 below for an equational presentation of complete join semilattices as $\mathcal{P}$–algebras). A $\mathcal{P}$–coalgebra can be considered as precisely a (notation of a) directed graph (i.e., a binary relation) on a set A: given $\alpha : A \to \mathcal{P}A$, then $\alpha(x)$ is the set of all elements related to x. Homomorphisms of $\mathcal{P}$–coalgebras are, however, more specialized than the usual graph homomorphisms: for coalgebras A_i expressed by graphs $R_i \subseteq A_i \times A_i$ (i= 1,2) a homomorphism of $\mathcal{P}$–coalgebras is a function $f : A_1 \to A_2$ such that (a) f is a graph homomorphism, i.e., xR_1y implies $f(x)R_2f(y)$ and (b) given elements $x \in A_1$ and $y \in A_2$ with $f(x)R_2y$ there exists $x' \in A_1$ with xR_1x' and $f(x') = y$.

(5) Denote by

$$\mathcal{P}_{fin} : \mathsf{Set} \to \mathsf{Set}$$

the finite–power–set functor. $\mathcal{P}_{fin}$–coalgebras are the directed graphs of finite out–degree, with homomorphisms as defined in (4) above.

(6) **Labelled transition systems**, important both in modal logic and in process algebra, are naturally formalized as coalgebras. For a given set L of labels, a labelled transition system is a set A (of states) together with a binary relation $\xrightarrow{l}$ on A for every label $l \in L$. Put

$$F = \mathcal{P}(- \times L) : \mathsf{Set} \to \mathsf{Set}.$$

Every labelled transition system A defines an F–coalgebra by

$$\alpha_A \colon A \to \mathcal{P}(A \times L),\ x \longmapsto \{(y,l) \in A \times L;\ x \xrightarrow{l} y\}.$$

This gives rise to an equivalence between $\mathsf{Coalg}(F)$ and the category of labelled transition systems and system morphisms defined analogously to example (4) above. If $L = 1$ we obtain the above case, $F = \mathcal{P}$, and sometimes speak about *nonlabelled transition systems.*

(7) k**–algebras** Denote by Vec_k the category of vector spaces over a field k and linear functions. Recall that a k–algebra is a vector space A together with a bilinear multiplication, i.e., a linear function $A \otimes A \to A$, which is associative, and a unit e (which we consider as the corresponding morphism $e \colon k \to A$) of the multiplication (see (Sweedler 1969)). Thus, k–algebras are F–algebras (satisfying additional axioms) of the functor $F \colon \mathsf{Vec}_k \to \mathsf{Vec}_k$ given by

$$FA = (A \otimes A) + k.$$

1.6 Remark The above functors F_Ω are called *polynomial.* They can be introduced, more generally, on every category C with finite products and arbitrary coproducts as the smallest class of endofunctors containing Id_{C} and closed under finite products and all coproducts.

2. Free algebras and cofree coalgebras

2.1 Example For polynomial endofunctors F_Ω on Set, the concept of free algebra $X^\sharp$ on a set X of generators is well known. We can describe it either recursively as $X^\sharp = \cup_{i<\omega} X_i^\sharp$ where

$$\begin{aligned} X_0^\sharp &= \Omega_0 + X \\ &= F_\Omega \emptyset + X \quad \text{terms of depths 0 are nullary operations and variables} \\ X_{i+1}^\sharp &= \{(\omega, t_0, \dots, t_{n-1}) \mid \omega \in \Omega_n, t_0, \dots, t_{n-1} \in X_i^\sharp\} + X \\ &= F_\Omega X_i^\sharp + X \quad \text{terms of depths i+1} \end{aligned}$$

Or directly: $X^\sharp$ is the algebra of all finite “properly” labelled trees. “Properly” means that a node with $n > 0$ children is labelled by an n-ary operation, and a leaf is labelled by a variable or a nullary operation. We have the universal arrow $\eta_X : X \to X^\sharp$, embedding X into $X^\sharp$.

2.2 Remark A *free F-algebra* on an object X (of “variables”) of C can be defined as a pair consisting of an F-algebra

$$FX^\sharp \xrightarrow{\varphi_X} X^\sharp \quad \text{and a morphism} \quad \eta_X : X \to X^\sharp$$

with the universal property that given an F-algebra (C, α_C) and a morphism $f \colon X \to C$ of C there exists a unique F-homomorphism $f^\sharp$ extending f, i.e., such that the following diagram

$$\begin{array}{ccccc} FX^\sharp & \xrightarrow{\varphi_X} & X^\sharp & \xleftarrow{\eta_X} & X \\ {\scriptstyle Ff^\sharp}\downarrow & & \downarrow{\scriptstyle f^\sharp} & \swarrow{\scriptstyle f} & \\ FC & \xrightarrow{\alpha_C} & C & & \end{array}$$

commutes. In other words, η_X is a universal arrow of the forgetful functor $U\colon \mathsf{Alg}(F) \to \mathsf{C}$.

In particular, if $X = 0$ (an initial object of C) then $0^\sharp$ is an initial F–algebra. Recall that by the famous *Lambek's Lemma* (Lambek 1968) $\alpha_{0^\sharp}\colon F0^\sharp \to 0^\sharp$ is an isomorphism.

2.3 Lemma *Let* C *have binary coproducts. An initial* $F(-)+X$*–algebra is equal to a free* F*–algebra on* X*.*

2.4 Corollary *For a free* F*–algebra* $X^\sharp$ *one has* $X^\sharp \simeq FX^\sharp + X$

This is Lambek's Lemma applied to $F_X = F(-) + X$.

It is good to have a name for endofunctors F such that every object of C admits a free F–algebra, that is, such that U has a left adjoint.

2.5 Definition ((Adámek and Trnková 1990)) An endofunctor $F\colon \mathsf{C} \to \mathsf{C}$ is called a *varietor* provided that a free F–algebra exists on every C–object.

2.6 Examples (1) Polynomial endofunctors on Set are varietors.

(2) The finite–power–set endofunctor of Set is a varietor. This follows from the fact that it is finitary, i.e., preserves filtered colimits. Also the countable–power–set endofunctor of Set is a varietor because it preserves $\aleph_1$–filtered colimits. We will see below that every λ–*accessible* endofunctor of Set, i.e., every functor preserving λ–filtered colimits (for some infinite cardinal λ) is a varietor.

(3) The power–set endofunctor $\mathcal{P}$ of Set is not a varietor since an initial $\mathcal{P}$–algebra does not exist.

(4) Generalizing (1) above let C have colimits and finite products such that colimits of ω–chains commute with finite products. Then every polynomial endofunctor F_Ω on C is a varietor. In fact, it is easy to see that, for all Ω, F_Ω preserves colimits of ω–chains. And then free algebras can be obtained by the following

2.7 Finitary Free-Algebra Construction (see (Adámek 1974)): This is an application of the famous construction of an *initial F-algebra* (the free F-algebra on 0, an initial object of C) as a colimit of the chain

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2 0} F^3 0 \cdots$$

to the functor $F_X = F(-) + X$ (see Lemma 2.3 above). Let C have countable colimits. Given an object X in C we define an ω–chain $X_i^\sharp (i < \omega)$ as follows:

$$0 \xrightarrow{!} F0 + X \xrightarrow{F!+X} F(F0+X)+X \xrightarrow{F(F!+X)+X} F(F(F0+X)+X)+X \cdots$$

That is

First step $X_0^\sharp = 0$, $X_1^\sharp = F0{+}X$ with $x_{0,1}^\sharp$ the unique morphism $0 \xrightarrow{!} F0 + X$,
Induction step $X_{i+1}^\sharp = FX_i^\sharp + X$ and $x_{i+1,j+1}^\sharp = Fx_{i,j}^\sharp + X$ for all $i \le j$.

Lemma *Let* C *be cocomplete and* F *preserve colimits of* ω*–chains. Then for every object* X *of* C,

$$X^\sharp = \operatorname*{colim}_{i<\omega} X_i^\sharp$$

is a free F*–algebra on* X.

More detailed: let $(X_i^\sharp \xrightarrow{x_i} X^\sharp)$ be a colimit cocone. Since $F(-) + X$ preserves that colimit we have a unique morphism

$$\varphi_X\colon FX^\sharp + X \to X^\sharp \quad \text{with} \quad \varphi_X \circ (Fx_i + X) = x_{i+1}$$

The two components $\eta_X\colon X \to X^\sharp$ and $\alpha_X\colon FX^\sharp \to X^\sharp$ of φ_X form a free F-algebra on X.

Proof. For every F-algebra (C, α_C) and any morphism ("assignment to variables") $f\colon X \to C$ define a cocone of the above chain (*computation of terms*) recursively as follows:

$$f_0^\sharp =! \quad \text{and} \quad f_{i+1}^\sharp = [\alpha_C \circ Ff_i^\sharp, f]$$

Then the (unique) factorization $X_i^\sharp \xrightarrow{x_i} X^\sharp \xrightarrow{f^\sharp} C = f_i^\sharp$ of that cocone gives the (unique) homomorphism $f^\sharp\colon (X^\sharp, \alpha_X) \to (C, \alpha_C)$ with $f = f^\sharp \circ \eta_X$. □

2.8 Examples (1) For the endofunctor $FY = Y \times Y + 1$ (i.e., one constant and one binary operation) on Set, we know that the terms in $X_i^\sharp$ are just the binary trees of depth $\le i$ with leaves labelled in $X+1$. This corresponds precisely to the construction above.

(2) For the endofunctor $FY = Y^{\mathbb{N}}$ (i.e., one ω-ary operation) on Set we again might form the sets $X_i^\sharp$ of terms, but here the colimit after ω steps does not give a free F-algebra, of course: we need ω_1 steps of the following

2.9 Free-Algebra Construction (see (Adámek 1974) or (Adámek and Trnková 1990, IV.3.2)): Let C be a cocomplete category. For every endofunctor F on C and every object X ("of variables") in C define a transfinite chain of objects $X_i^\sharp$ (i any ordinal) and connecting morphisms

$$x_{i,j}^\sharp\colon X_i^\sharp \to X_j^\sharp \ (i \le j)$$

by the following transfinite induction:

First step $X_0^\sharp = 0$, $X_1^\sharp = F0{+}X$ with $x_{0,1}^\sharp$ the unique morphism $0 \xrightarrow{!} F0 + X$,
Isolated step $X_{i+1}^\sharp = FX_i^\sharp + X$ for all ordinals i, $x_{i+1,j+1}^\sharp = Fx_{i,j}^\sharp + X$ for all $i \le j$,
Limit step $X_j^\sharp = \mathsf{colim}_{i<j} X_i^\sharp$ for all limit ordinals j with colimit cocone $x_{i,j}^\sharp$, $i < j$.

We say that the free algebra construction *stops* after k steps provided that $x_{k,k+1}^\sharp$ is an isomorphism.

2.10 Lemma *Let* C *be cocomplete and* F *preserve colimits of* λ*–chains. Then for every object* X *of* C, $X^\sharp_\lambda$ *is a free* F*–algebra on* X.

This is analogous to Lemma 2.7.

2.11 Computation of terms In classical universal algebra, where $X^\sharp_i$ are the terms of depth $< i$ over variables from X, we have the concept of computation of terms in an algebra A: for every interpretation of variables, i.e., every function $f\colon X \to A$ we extend f to a cocone $f^\sharp_i\colon X^\sharp_i \to A$ of the free–algebra construction by the well–known algebraic recursion. This is a special case of the following:

Let F be an endofunctor of a cocomplete category C. For every F–algebra (A, α_A) and every morphism $f\colon X \to A$ in C we define a cocone $f^\sharp_i\colon X^\sharp_i \to A$ of the free–algebra construction as follows:

First step $f^\sharp_0\colon 0 \to A$ is the unique morphism.

Isolated step Given $f^\sharp_i\colon X^\sharp_i \to A$ let

$$f^\sharp_{i+1}\colon FX^\sharp_i + X \to A$$

have components $FX^\sharp_i \xrightarrow{Ff^\sharp_i} FA \xrightarrow{\alpha_A} A$ and $X \xrightarrow{f} A$. Observe that the compatibility condition

$$f^\sharp_i = f^\sharp_{i+1} \circ x^\sharp_{i,i+1}$$

is fulfilled (an easy proof by transfinite induction).

Limit Step Given a limit ordinal i and given the cocone $f^\sharp_j\colon X^\sharp_j \to A$, $j < i$, of the first i steps of the free–algebra construction, there is a unique

$$f^\sharp_i\colon \operatorname*{colim}_{j<i} X^\sharp_j \to A$$

with the (obligatory) property $f^\sharp_j = f^\sharp_i \circ x^\sharp_{ji}$ for all $j < i$.

2.12 Lemma *Homomorphisms of* F*–algebras preserve computation of terms, i.e., given a homomorphism* $h\colon (C, \alpha_C) \to (D, \alpha_D)$ *and an assignment of variables* $f\colon X \to C$ *then, for all ordinals* i, $(h \circ f)^\sharp_i = h \circ f^\sharp_i$.

Proof. We proceed by transfinite induction on i. The case $i = 0$ is trivial since the domain of both sides is 0. The limit case is immediate from the universal property of colimits. In the isolated case we assume the above equation for i and derive

$$\begin{aligned}
(h \circ f)^\sharp_{i+1} &= [\alpha_D \circ F(h \circ f)^\sharp_i,\ h \circ f] \\
&= [\alpha_D \circ Fh \circ Ff^\sharp_i,\ h \circ f] && \text{by induction hypothesis} \\
&= [h \circ \alpha_C \circ Ff^\sharp_i,\ h \circ f] && \text{since } h \text{ is a homomorphism} \\
&= h \circ [\alpha_C \circ Ff^\sharp_i,\ f] \\
&= h \circ f^\sharp_{i+1}
\end{aligned}$$

□

2.13 Proposition *If the free–algebra construction stops after* k *steps, then* $X^\sharp_k$ *is a free* F*–algebra on* X.

More detailed: Denoting the components of $x_{k,k+1}^{-1} : FX_k^\sharp + X \to X_k^\sharp$ by

$$\alpha_X : FX_k^\sharp \to X_k^\sharp \text{ and } \eta_X : X \to X_k^\sharp$$

respectively, these form a free F–algebra on X.

Proof. Let (C, α_C) be an F–algebra $f : X \to C$ a morphism. From the equation

$$f_k^\sharp = f_{k+1}^\sharp \circ x_{k,k+1} = [\alpha_C \circ Ff_k^\sharp, f] \circ [\alpha_X, \eta_X]^{-1}$$

it follows that $f_k^\sharp$ is a homomorphism extending f. The uniqueness is easy to verify. □

2.14 Definition ((Adámek and Trnková 1990)) A functor $F : \mathsf{C} \to \mathsf{C}$ with C cocomplete is called *constructive varietor* provided that its free algebra construction stops for each object X in C.

2.15 Remark A functor F is said to be *accessible* if it preserves, for some infinite cardinal λ, all λ–filtered colimits (we speak more precisely of λ–accessiblity, then). Accessible Set-functors are precisely the bounded functors of (Kawahara and Mori 2000) as proved in (Adámek and Porst 2001). Since colimits of λ–chains are λ–filtered, we get the following

Corollary *Every accessible endofunctor of a cocomplete category is a constructive varietor.*

2.16 A full characterization of endofunctors of Set which are varietors has been presented in (Adámek and Trnková 1990). Let us call an endofunctor *trivial* provided that its restriction to all nonempty sets and functions is constant. (E.g., the functor $\emptyset \mapsto \emptyset$ and $X \mapsto 1$ for all $X \neq \emptyset$ is trivial.) An object X such that FX is isomorphic to X is called a *fixed point* of F.

Theorem *The following conditions on an endofunctor F of* Set *are equivalent:*

(i) F is a varietor,

(ii) F is a constructive varietor,

(iii)F has arbitrarily large fixed points, or F is trivial.

For each such functor F the category of F–algebras is cocomplete.

Proof. See (Adámek and Trnková 1990, IV.43 and IV.8.2). □

Remark An analogous result holds for Vec_k: (constructive) varietors are precisely the endofunctors which have arbitrarily large fixed points, or are constant, see (Adámek and Trnková 1990, IV.5.5).

2.17 Example of a varietor on Set which is not accessible. Assume the generalized continuum hypothesis. For a class M of cardinals with $0, 1 \in M$ denote by

$$\mathcal{P}_M : \mathsf{Set} \to \mathsf{Set}$$

the functor assigning to every set A the set of all subsets $B \subseteq A$ with $\mathsf{card} B \in M$. $\mathcal{P}_M$ assigns to a morphism $f : A \to A'$ the function defined as follows:

$B \mapsto f[B]$ if f restricted to B is one–to–one, else $B \mapsto \emptyset$.

It is easy to see that $\mathcal{P}_M$ is accessible iff M is a set. And an infinite cardinal α is a fixed point of $\mathcal{P}_M$ iff $\alpha \notin M$. (The proof is easy: use the fact that $\alpha^\beta = \alpha$ for all infinite cardinals $\beta < \alpha$, see (Aczel et al. 2002).)

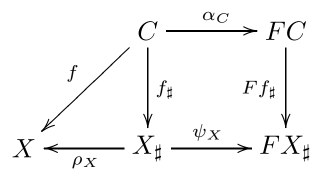
Now choose a class M of cardinals such that neither M nor its complement $\mathsf{card} \setminus M$ is a set. Then $\mathcal{P}_M$ is a varietor because it has arbitrarily large fixed points (namely all whose cardinality does not lie in M), but it is not accessible.

2.18 Example of a (trivial) non–constructive varietor: let C be the usual ordered class of all ordinals extended by a largest element, $\top$, considered as a category (with $\vee$ as binary coproduct). The functor $F\colon \mathsf{C} \longrightarrow \mathsf{C}$ with

$$F(i) = i + 1 \text{ for all ordinals } i \text{ and } F(\top) = \top$$

is a varietor: $\mathsf{Alg}(F)$ consists of a single algebra $(\top, \alpha_\top)$. However, for $X = 0$ we get $X_i^\sharp = i$ for every ordinal i, thus, the free–algebra construction never stops.

2.19 Cofree coalgebras are the corresponding dualization of free algebras. A cofree F-coalgebra on an object X ("of colours") of C is a coalgebra $\psi_X : X_\sharp \to FX_\sharp$ together with a ("colouring") morphism $\rho_X : X_\sharp \to X$ having the universal property that given an F-coalgebra (C, α_C) and a morphism $f\colon C \to X$ of C there exists a unique F-coalgebra homomorphism $f_\sharp$, such that the following diagram commutes:



In other words, ρ_X is a couniversal arrow of the forgetful functor $U\colon \mathsf{Coalg}(F) \to \mathsf{C}$.

2.20 Definition An endofunctor $F\colon \mathsf{C} \to \mathsf{C}$ is called a *covarietor* provided that a cofree F-coalgebra exists on every C-object.

This terminology is justified by the following remark based on 1.4 and 2.3.

2.21 Remark The following are equivalent for any $F\colon \mathsf{C} \to \mathsf{C}$:

(i) F is a covarietor.

(ii) F^{op} is a varietor.

In case C has finite products, another equivalent condition is:

(iii) For every object X in C the functor $F^X = F(-) \times X$ has a final (= terminal) coalgebra.

2.22 Example Every accessible endofunctor of Set (in fact of any locally presentable category) is a covarietor as shown in (Barr 1993).

A dualization of the free-algebra construction above gives the following

2.23 Cofree Coalgebra Construction: Let C be a complete category. For every endofunctor F on C and every object X ("of colours") in C define a transfinite cochain of objects $X^i_\sharp$ (i any ordinal) and connecting morphisms $x^{i,j}_\sharp : X^i_\sharp \to X^j_\sharp$ $(i \geq j)$ as follows (where 1 denotes a terminal object of C):

First step $\quad X^0_\sharp = 1,\ X^1_\sharp = F1 \times X \quad$ with $\ x^{1,0}_\sharp : F1 \times X \xrightarrow{!} 1\ $ the unique morphism,
Isolated step $\quad X^{i+1}_\sharp = FX^i_\sharp \times X \quad$ for all ordinals i, $\ x^{i+1,j+1}_\sharp = Fx^{i,j}_\sharp \times X$ for all $i \geq j$,
Limit step $\quad X^j_\sharp = \mathsf{lim}_{i<j} X^i_\sharp \qquad$ for all limit ordinals j with limit cone $x^{j,i}_\sharp,\ i < j$.

If this cochain construction *stops after k steps*, i.e, $x^{k+1,k}_\sharp : FX^k_\sharp \times X \to X^k_\sharp$ is an isomorphism, for some ordinal k, then $X^k_\sharp$ is a cofree F-coalgebra on X. More detailed: Denoting the components of the inverse of $x^{k+1,k}_\sharp$ by

$$\alpha_X : X^k_\sharp \to FX^k_\sharp \quad \text{and} \quad \rho_X : X^k_\sharp \to X$$

these form a cofree F-coalgebra on X. For an F-coalgebra (C, α_C) and a morphism $f : C \to X$ the extension $f_\sharp$ of f is the k-th member of the cocone $f^i_\sharp : C \to X^i_\sharp$ which is defined dually to 2.11.

2.24 Definition A functor $F : \mathsf{C} \to \mathsf{C}$ is called *constructive covarietor* provided that the above Construction 2.23 stops for each C-object X.

2.25 Examples (1) Let $F = \mathrm{Id}_{\mathsf{Set}}$. For every set X (of colours) we form the first steps of the cofree–coalgebra construction:

$$1 \leftarrow 1 \times X \leftarrow 1 \times X \times X \leftarrow 1 \times X \times X \times X \leftarrow \dots$$

where each of the connecting maps is the projection (forgetting the last component). A limit of this ω^{op}–sequence is $X_\sharp = X^\omega$, the set of all infinite words over X (with projections as limit maps).
For a given unary (co–)algebra (A, α_A) and a given colouring $f : A \to X$ the unique homomorphism $f_\sharp : A \to X_\sharp$ assigns to every element $a \in A$ the word of colours of the elements $a, \alpha_A a, \alpha_A^2 a, \dots$

(2) For polynomial endofunctors F_Ω of Set a description of a final coalgebra $1_\sharp$ (which also can be derived from the cofree coalgebra construction) is as follows: the elements of the coalgebra $1_\sharp$ are finite and infinite Ω–labelled trees (where Ω–labelled means that a node with k children is labelled by an element of Ω_k for all $k = 0, 1, 2 \dots$). The structure map

$$\alpha_{1_\sharp} : 1_\sharp \to \sum_{k \in \mathbb{N}} \Omega_k \times (1_\sharp)^k$$

assigns to a tree t whose root is labelled by $\omega \in \Omega_k$ the k–tuple of the subtrees given by the sons of the root (i.e., the corresponding element of $(1_\sharp)^k$ in the summand indexed by ω); i.e., the coalgebra structure is the inverse of tree tupling.
Due to 2.4 we obtain immediately a description of $X_\sharp$ for any set X (of colours). Since $F_\Omega(-) \times X \simeq F_{\Omega^X}(-)$ for the signature $\Omega^X = (\Omega_n \times X)_{n \in \mathbb{N}}$, we obtain the following:

$X_\sharp$ is the coalgebra of finite and infinite trees with a node having k children labelled in $\Omega_k \times X$ for all $k = 0, 1, 2, \ldots$.

(3) For $\mathcal{P}_{fin}$ the final–coalgebra construction stops after $\omega + \omega$ steps, see (Worell 2000).

2.26 Theorem *Every covariator in* Set *is constructive.*

In fact, to prove this, it is, due to Remark 2.21, sufficient to prove that whenever F has a final coalgebra, the cofree–coalgebra construction for 1 stops, see (Adámek and Koubek 1995).

2.27 A full characterization of covarietors F in Set is not known. Let us call a cardinal number α an *exponential fixed point* of F provided that every set X whose cardinality lies between α and 2^α is a fixed point of F.

2.28 Theorem ((Adámek and Koubek 1995)) *Every endofunctor of* Set *with arbitrarily large exponential fixed points is a covarietor.*

2.29 Theorem ((Worell 2000)) *Every λ–accessible endofunctor of* Set *is a covarietor, and the cofree–coalgebra construction stops in $\lambda + \lambda$ steps.*

2.30 Example of a covarietor on Set which is not accessible. Assume the Generalized Continuum Hypothesis. Let M be a class of cardinals containing $0, 1$ such that M is not a set, but there exist arbitrarily large cardinals α with $M \cap [\alpha, 2^\alpha] = \emptyset$. Then $\mathcal{P}_M$ (see 2.17) is a covarietor because it has arbitrarily large exponential fixed points. However, $\mathcal{P}_M$ is not accessible.

3. Varieties and Covarieties

In classical finitary algebra, an *equation* in variables from a set X is a pair of terms, i.e., elements of a free algebra $X^\sharp$ on X. That is, a parallel pair

$$u, v\colon 1 \to X^\sharp$$

of morphisms in Set. An algebra A *satisfies* that equation provided that for every interpretation of variables in A, i.e., every function $f\colon X \to A$, the unique homomorphism $f^\sharp\colon X^\sharp \to A$ extending f (i.e., the function computing terms in A) satisfies

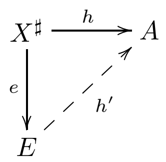
$$f^\sharp u = f^\sharp v.$$

This is equivalent to saying that $f^\sharp$ factorizes through a coequalizer

$$e\colon X^\sharp \to E$$

of u and v.

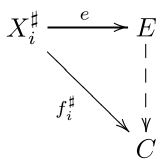
Thus, every equation gives us a regular quotient $e\colon X^\sharp \to E$ of the free algebra $X^\sharp$ defining satisfaction in A by requesting that every homomorphism $h\colon X^\sharp \to A$ factors through e:



Conversely, given a regular quotient $e\colon X^\sharp \to E$, then satisfaction of e in the above sense is equivalent to the satisfaction of the "classical" equations $u = v$ for all pairs, $u, v\colon 1 \to X^\sharp$ with $eu = ev$. This leads us to the following concepts, closely related to those introduced in (Adámek 1983) and (Banaschewski and Herrlich 1976):

3.1 Definition Let F be an endofunctor of a cocomplete category C. Using the notation $X_i^\sharp$ and $f_i^\sharp$ as in 2.9 and 2.11 we define:

1. An *equation arrow over* X is a regular epimorphism $e\colon X_i^\sharp \to E$ for some ordinal i. An F-algebra (C, α_C) is said to *satisfy* e provided that for every morphism $f\colon X \to C$ the morphism $f_i^\sharp$ factors through e:



2. For any class $\mathcal{E}$ of equation arrows, $\mathsf{Alg}(F, \mathcal{E})$ denotes the full subcategory of $\mathsf{Alg}(F)$ formed by all F-algebras satisfying every $e \in \mathcal{E}$. Such categories are called *equational categories (of F-algebras)* over C.

3.2 Examples (1) Varieties of finitary algebras are precisely the equational categories

$$\mathsf{Alg}(\mathsf{F}_\Omega, \mathcal{E}).$$

Here equations $e\colon X_i^\sharp \to E$ can always be substituted by equations $e\colon X^\sharp \ (= X_\omega^\sharp) \to E$, as we see below. Moreover, for every variety we only need a single equation–morphism $e\colon X^\sharp \to E$ with X a countable set: just consider the smallest equivalence $\sim$ on $X^\sharp$ with $u \sim v$ for every equation $u = v$ in the presentation of the variety. This is generalized in 3.8 below.

(2) $\mathsf{Alg}(F)$ is, of course, an equational class for any functor F: put $\mathcal{E} = \emptyset$.

(3) If C has coequalizers, then for every parallel pair

$$u, v\colon C \to X_i^\sharp \quad (i \in \mathsf{Ord})$$

of morphisms a coequalizer of u and v is an equational arrow. An F–algebra A satisfies this arrow iff

$$(*) \qquad f_i^\sharp u = f_i^\sharp v \quad \text{for all} \quad f\colon C \to A$$

Conversely, if C has kernel pairs, then every equational arrow $e\colon X_i^\sharp \to E$ is a coequalizer of its kernel pair $u, v\colon C \to X_i^\sharp$, thus, satisfaction can be defined via (*). We use the usual notation $u = v$ for the equation morphism that is the coequalizer of u and v.

3.3 Remark We want to distinguish between equational classes of algebras and varieties. The latter are requested to have the (important) property that free algebras exist on every object C of variables. This is not automatic, e.g., $\mathsf{Alg}(\mathcal{P})$, for the power–set functor $\mathcal{P}$, is an equational class but not a variety.

Definition By a *variety* of F–algebras is meant an equational class $\mathsf{Alg}(F, \mathcal{E})$ such that the forgetful functor into C (a restriction of U_F) has a left adjoint.

3.4 Example "Classical" finitary varieties are precisely varieties of F–algebras for polynomial endofunctors F_Ω of Set.

3.5 Example (complete semilattices) The category of complete join semilattices has as objects all posets with joins and as morphisms all functions preserving joins. The objects can be described as precisely those $\mathcal{P}$–algebras $\alpha_A \colon \mathcal{P}A \to A$ for which the (join–) operation α_A satisfies

i. $\alpha_A\{a\} = a$ for every $a \in A$

and

ii. $\alpha_A\{\alpha_A(A_i) \mid i \in I\} = \alpha_A \bigcup\limits_{i \in I} A_i$ for all collections $A_i \subseteq A,\ i \in I$.

We can express (i) and (ii) by equation–arrows as follows.

ad i. Put $X = \{x\}$ and observe that we have

$$\begin{aligned} X_0^\sharp &= \emptyset \\ X_1^\sharp &= \mathcal{P}\emptyset + X = \{\emptyset, x\} \\ X_2^\sharp &= \mathcal{P}\{\emptyset, x\} + X = \{\emptyset, x, \{x\}, \{\emptyset\}, \{\emptyset, x\}\} \end{aligned}$$

Condition (i) is expressed by the equation

$$\{x\} = x$$

or, more precisely, by the equation–arrow $e \colon X_2^\sharp \to X_2^\sharp/{\sim}$ which is the canonical map of the equivalence $\sim$ whose only nonsingleton equivalence class is $\{x, \{x\}\}$.

In fact, a $\mathcal{P}$–algebra (A, α_A) satisfies $x = \{x\}$ iff for every morphism $f \colon X \to A$ (i.e., every element $a = f(x)$ of A) the extension $f_2^\sharp \colon X_2^\sharp \to A$ satisfies $f_2^\sharp(\{x\}) = f_2^\sharp(x)$. Equivalently: $\alpha_A\{a\} = a$.

ad ii. Put $X = A$ and observe that $\bigcup\limits_{i \in I} A_i$ is an element of $X_2^\sharp$ and $\{A_i \mid i \in I\}$ an element of $X_3^\sharp$. Consider the equation

$$\{A_i \mid i \in I\} = \bigcup_{i \in I} A_i,$$

or, more precisely, the equation–arrow $\bar{e} \colon X_3^\sharp \to X_3^\sharp/{\sim}$ where $\sim$ is the equivalence whose only nonsingleton equivalence class is $\{\{A_i \mid_{i \in I}\}, \bigcup\limits_{i \in I} A_i\}$. The satisfaction of the equation arrow $\bar{e}$ implies (ii) trivially: just use $f = \mathrm{id} \colon X \to A$. Conversely, every $\mathcal{P}$–algebra satisfying (i) and (ii) above satisfies the equation arrow $\bar{e}$.

Remark Free complete semilattices exist: they are the powerset–semilattices. Thus complete semilattices form a variety of $\mathcal{P}$–algebras.

3.6 Lemma *For arbitrary ordinals $i \leq j$, any equation*

$$u = v \quad \textit{with} \quad u, v \colon C \to X_i^\sharp$$

is equivalent to the equation

$$x_{ij}u = x_{ij}v.$$

That is, an F–algebra satisfies the first one iff it satisfies the latter.

Proof. This follows from $f_j^\sharp x_{ij} = f_i^\sharp$, see 2.12. □

3.7 Corollary *If F is a constructive varietor, we only need to consider equations $u = v$ for morphisms $u, v \colon C \to X^\sharp$.*

3.8 Corollary *For every accessible endofunctor F of* Set *and any equational class* V *of F-algebras there is a single equation arrow e presenting* V*, i.e. such that* $\mathsf{V} = \mathsf{Alg}(F, \{e\})$.

In fact, if F is λ-accessible, then it is easy to see that every equation $u = v$ for elements $u, v \in X^\sharp$ (see 3.7 and 2.15) is equivalent to an equation $u' = v'$ for suitable elements $u', v' \in M^\sharp$, where M is an (arbitrary) set of cardinality λ. Thus, every equational class is presented by a set pairs of elements of $M^\sharp$. For each of these pairs form a coequalizer, and obtain a set of quotients of $M^\sharp$ in Set. Forming a pushout, e, of that set of (regular) epimorphisms, we obtain $\mathsf{V} = \mathsf{Alg}(F, \{e\})$.

By formally dualizing Definition 3.1, see Lemma 1.4, we obtain the following

3.9 Definition Let F be an endofunctor of a complete category C.

1. A *coequation arrow over X* is a regular monomorphism $m \colon M \to X_\sharp^i$ for some ordinal i. An F–coalgebra (C, α_C) is said to *satisfy* m provided that for every morphism $f \colon C \to X$ the morphism $f_\sharp^i$ factors through m.
2. For any class $\mathcal{M}$ of coequation arrows $\mathsf{Coalg}(F, \mathcal{M})$ denotes the full subcategory of $\mathsf{Coalg}(F)$ formed by all F–coalgebras satisfying every $m \in \mathcal{M}$. Such categories are called *coequational categories (of F–coalgebras)* over C.

3.10 Notation For $\mathsf{C} = \mathsf{Set}$, given an element $d \in X_\sharp^i$ we denote by $\neg d$ the coequation arrow

$$X_\sharp^i \setminus \{d\} \hookrightarrow X_\sharp^i.$$

Thus, a coalgebra A satisfies $\neg d$ iff for every colouring $f \colon A \to X$ we have

$$f_\sharp^i(a) \neq d \text{ for all } a \in A.$$

Observe that we do not need any more general coequations: instead of a coequation arrow $m \colon M \to X_\sharp^i$ we can always consider the coequations $\neg d$ for all $d \in X_i^\sharp \setminus m[M]$.

3.11 Example (nonlabelled transition systems (= graphs)) Here $F = \mathcal{P}$, the power–set functor. We give some examples of covarieties.

(a) **No terminals.** A state s of a transition system A is *terminal* if $s \to t$ holds for no state t. The non-existence of terminal states is expressed by the coequation $\neg\emptyset$ for $\emptyset \in \mathcal{P}1 = 1^1_\sharp$. In fact, we have $f^0_\sharp\colon A \to 1$, the constant function, and $f^1_\sharp\colon A \to \mathcal{P}1 = \{\emptyset, 1\}$ taking value $\emptyset$ for terminal states and 1 for nonterminal states.

(b) **One–step reachability of terminals.** Consider

$$1^2_\sharp = \mathcal{P}\mathcal{P}1 = \{\emptyset, \{\emptyset\}, \{1\}, \{\emptyset, 1\}\}.$$

Given a state s then $f^2_\sharp(s) = \emptyset$ iff s is terminal, $f^2_\sharp(s) = \{\emptyset\}$ iff s is nonterminal but only terminal states can be reached from s directly, and $f^2_\sharp(s) = \{1\}$ iff s is nonterminal and only nonterminal states can be reached from s directly. Thus, the coequation $\neg\{1\}$ means: from every state a terminal state is reachable in at most 1 step.

(c) **Reachability of terminals.** In the cofree–coalgebra sequence

$$1 \leftarrow \mathcal{P}1 \leftarrow \mathcal{P}\mathcal{P}1 \leftarrow \dots$$

consider the elements $d_n \in \mathcal{P}^n 1$ where d_0 is the unique element of 1, and

$$d_{n+1} = \{d_n\}.$$

The sequence $d = (d_n) = (d_0, 1, \{1\}, \{\{1\}\} \dots)$ is compatible, i.e., is an element of the limit

$$1^\omega_\sharp = \lim_{n<\omega} 1^n_\sharp.$$

For a state s we have $f^\omega_\sharp(s) = d$ iff $f^1_\sharp(s) = \{1\}$ (i.e., only nonterminals are reached in 1 step) and $f^2_\sharp(s) = \{1\}$ (i.e., only nonterminals are reached in 2 steps) etc. Thus, $\neg d$ expresses the fact that from each state a terminal state is reachable.

(d) **Reflexivity**, i.e., the fact that $s \to s$, for each state s, can be easily expressed using two colours. Put $2 = \{0, 1\}$ and consider the cofree–coalgebra sequence

$$1 \leftarrow \mathcal{P}(1) \times 2 \leftarrow \mathcal{P}\left(\mathcal{P}(1) \times 2\right) \times 2 \leftarrow \dots$$

Given a colouring $f\colon A \to \{0, 1\}$, then $f^2_\sharp(s) = (M, f(s))$ where $M \subseteq \mathcal{P}(1) \times 2$ is the set of all pairs (B, i) where, for some state s' reachable from s, we have $f(s') = i$ and, either $B = \emptyset$ if s' is a terminal state, or $B = 1$, else.

Suppose that $f^2_\sharp(s) = (\{1, 0\}, 1)$, then s has colour 1 but every reachable state has colour 0. Thus, s is not reflective. It is easy to see that the coequation $\neg(\{1, 0\}, 1)$ means: for every state s either $s \to s$, or s is terminal. Thus, reflexivity is axiomatized by the two coequations $\neg\emptyset$ and $\neg(\{1, 0\}, 1)$.

(e) **Symmetry**, i.e., the fact that

$$s \to t \text{ implies } t \to s \quad \text{(for all states } s, t).$$

Here we use

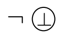
$$f^3_\sharp\colon A \to \mathcal{P}\left(\mathcal{P}\left(\mathcal{P}(1) \times 2\right) \times 2\right) \times 2$$

given by $f^3_\sharp(s) = (R, f(s))$ where $R \subseteq \mathcal{P}\left(\mathcal{P}(1) \times 2\right) \times 2$ is the set of all $f^2_\sharp(t)$ for edges $s \to t$. Suppose that for a colouring $f \colon A \to \{0,1\}$ and a state s we have $f^2_\sharp(s) = (R, 0)$ where $R \subseteq \mathcal{P}(\mathcal{P}(1) \times 2) \times 2$ contains an element of the form $(-, 1)$. Every element $r \in R$ (a pair consisting of a subset of $\mathcal{P}(1) \times 2$ and a colour) corresponds to a state t with $s \to t$ in the sense that $f^1_\sharp(t) = r$. Thus, if some element $r \in R$ has the form $r = (Q, 1)$, then we have a state t directly reachable from s of colour 1 such that $f^2_\sharp(t) = Q$. Symmetry then guarantees that Q contains $(1, 0)$: in fact, we have the edge $t \to s$ where $f^1_\sharp(s) = (1, 0)$. Thus, in a symmetric transition system $f^3_\sharp(s)$ never has the form $(R, 0)$ where some element of R is $(Q, 1)$ with $(1, 0) \notin Q$. Symmetry is axiomatized by the following coequations

$$\neg(R, 0) \quad \text{for all} \;\; R \subseteq \mathcal{P}(\mathcal{P}(1) \times 2) \times 2 \text{ containing some } \;(Q, 1) \text{ with } \;(1, 0) \notin Q.$$

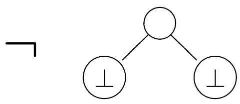
3.12 Example (dynamical systems on two inputs) Here we consider the functor $FX = X \times X + 1$ of Example 1.5(1), thus, coalgebras are dynamical systems on two inputs with deadlock states. Recall that the final coalgebra consists of binary trees with leaves labelled by $\bot$.

(a) The coequation



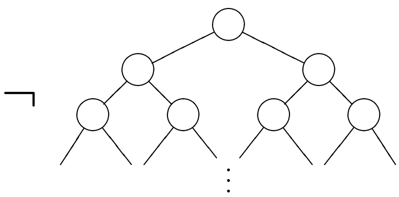
means: no deadlock states.

(b) The coequation



means: every non–deadlock state can continue indefinitely (without encountering a deadlock state).

(c) The coequation

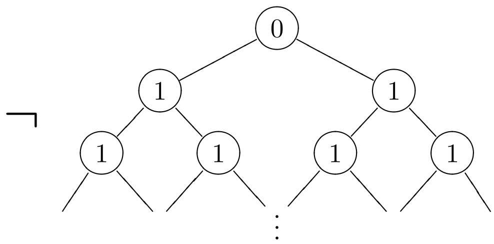


means: from every state a deadlock state is reachable.

(d) For $2 = \{0, 1\}$ we have

$2^\sharp$ = coalgebra of binary trees with leaves labelled by $(\bot, 0)$ or $(\bot, 1)$
and internal nodes labelled by 0 or 1.

The coequation



means: from every non–periodic state a deadlock state is reachable. (A state is periodic if the system can return to that state after $n > 0$ steps.)

3.13 Example (Dynamical system on one input [without deadlock]) Here $F = \mathrm{Id}_{\mathsf{Set}}$, and coalgebras are just unary algebras. We have $X_\sharp = X^\omega$, the set of all infinite words on X.

Given an element a of a (co)algebra A, we say that a *lies over a cycle* of length n provided that n is the smallest number with $\alpha_A^k(a) = \alpha_A^{k+n}(a)$ for some $k \in \mathbb{N}$. The smallest such k is called the *distance* of a from the cycle.

(a) For $2 = \{0, 1\}$ consider

$$d = 010101\ldots = (01)^\omega.$$

An algebra A satisfies the coequation $\neg d$ iff every element lies over an odd cycle. In fact, A satisfies $\neg d$ iff for every colouring $f\colon A \to \{0,1\}$, given an element $a \in A$, then the word

$$(f(a), f(\alpha_A(a)),\ f(\alpha_A^2(a))\ldots)$$

is distinct from d. This is certainly true if a lies over an odd cycle. Conversely, if a lies over an even cycle, or does not lie over any cycle (i.e., the elements $\alpha_A^k(a)$ are pairwise distinct), we can certainly choose a colouring $f\colon A \to \{0,1\}$ with

$$f(\alpha_A^k(a)) = \begin{cases} 0 & k \quad \text{even} \\ 1 & k \quad \text{odd.} \end{cases}$$

Then $f_\sharp(a) = 010101\ldots$.

(b) For

$$d = \underbrace{00\ldots0}_{s}\,010101\ldots = 0^s(01)^\omega$$

an algebra A satisfies $\neg d$ iff every element either lies over an odd cycle, or it lies over an even one and has distance less than s from it. Shortly: no element has distance s from an even cycle, and every element lies over a cycle.

(c) Analogously, for

$$d = \underbrace{00\ldots0}_{s}\,\underbrace{00\ldots0}_{p}\,1\,\underbrace{00\ldots0}_{p}\,1\ \ldots = 0^s(0^p1)^\omega$$

an algebra satisfies $\neg d$ iff no element has distance s from a cycle whose length is a multiple of p, and every element lies over a cycle.

Remark The above example shows that for every set $S \subseteq \mathbb{N} \times \mathbb{N}$ we have the following covariety C_S of unary algebras:

$A \in \mathsf{C}_S$, iff given $(s,p) \in S$, then no element of A has distance s from a cycle of length in $p\mathbb{N}$, and every element lies over a cycle.

One can show that these are precisely all covarieties in $\mathsf{Coalg}(Id)$.

4. Limits and Colimits

In this section we summarize results on limits and colimits of (co)algebras needed for the proof of Birkhoff's Variety Theorem in the subsequent section. Most of these results can be found in the existing literature, although e.g. the Colimit Theorem below seems to be new (and easy).

4.1 Products of F–algebras are easy to construct: suppose C has products and let (A_i, α_i) be a collection of F–algebras ($i \in I$). Form a product $A = \prod_{i \in I} A_i$ in C with projections $\pi_i : A \longrightarrow A_i$. Then there exists a unique $\alpha : FA \to A$ turning each π_i into a homomorphism, viz, $\alpha = \langle \alpha_i \circ F\pi_i \rangle : FA \to \prod A_i$. And it is easy to see that (A, α) is a product of the given F–algebras in $\mathsf{Alg}(F)$.

4.2 Limits of F–algebras The above construction is a special case of the following fact: the forgetful functor

$$U : \mathsf{Alg}(F) \to \mathsf{C}$$

creates limits. This means that, given a diagram D in $\mathsf{Alg}(F)$ and forming a limit cone $(A \xrightarrow{\pi_i} UD_i)$ of UD in C, there exists a unique $\alpha : FA \to A$ turning each π_i into a homomorphism. And (A, α) is, then, a limit of D. The proof is easy, see e.g. (Adámek and Trnková 1990).

Corollary *If C is complete then categories of F–algebras are complete.*

4.3 Corollary *A homomorphism in $\mathsf{Alg}(F)$ is an isomorphism (or a monomorphism) iff its underlying morphism is an isomorphism (or a monomorphism resp.) in C.*

If C has kernel pairs and F preserves coequalizers of kernel pairs, a homomorphism in $\mathsf{Alg}(F)$ is a regular epimorphism iff its underlying morphism is a regular epimorphisms in C.

In fact, a functor creating pullbacks preserves and reflects monomorphisms, isomorphisms, and kernel pairs. And regular epimorphisms are precisely the coequalizers of their kernel pairs.

Since the underlying functor of $\mathsf{Alg}(F)$ allows for unique transport of structures and every C-object carries only a set of F-algebras we get

4.4 Corollary *If C is wellpowered (i.e., every object has only a set of subobjects) then so are the categories of F–algebras.*

4.5 Colimits of F–algebras This is a more difficult topic. Firstly, colimits need not exist, in general. E.g., the power–set functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ does not have any fixed point. Thus, by Lambek's Lemma, an initial $\mathcal{P}$–algebra (i.e., the empty colimit in Alg $(\mathcal{P})$) does not exist.

One easy case is that of colimit types preserved by F:

4.6 Proposition *For every type of colimits in* C *which* F *preserves the category of* F*–algebras has colimits of that type, created by the forgetful functor.*

Proof. Given a diagram $D\colon \mathsf{D} \to \mathsf{Alg}(F)$ such that C has and F preserves colimits over D, form a colimit $(A_d \xrightarrow{c_d} A)_{d\in\mathsf{D}}$ of UD in C, where $UDd = A_d$, and observe that, since $(FA_d \xrightarrow{Fc_d} FA)_{d\in\mathsf{D}}$ is a colimit of FUD, there exists a unique $\alpha\colon FA \to A$ turning each c_d into a homomorphism. It is easy to see that (A,α) together with c_d $(d \in \mathsf{D})$ is a colimit of D in $\mathsf{Alg}(F)$. □

4.7 Example The polynomial functors $F_\Omega\colon \mathsf{Set} \to \mathsf{Set}$ preserve filtered colimits. This is true because (i) each functor $X \mapsto X^n$ (n a natural number) preserves filtered colimits, since finite products and filtered colimits commute in Set and (ii) F_Ω is a coproduct of functors (i), and coproducts commute with (filtered) colimits.

Therefore, filtered colimits of "classical" algebras are created by the forgetful functor.

4.8 Recall that a category C is said to have *regular factorizations* provided that every morphism of C factorizes as a regular epimorphism followed by a monomorphism. A functor $U\colon \mathsf{A} \to \mathsf{C}$ is said to *create* regular factorizations provided that for every morphism $f\colon A \to B$ in A with a regular factorization $Uf = m\circ e$ in C there exist unique morphisms m_0 and e_0 in A with $m = Um_0$ and $e = Ue_0$, and moreover, m_0 is a monomorphism and e_0 is a regular epimorphism with $f = m_0 \circ e_0$.

4.9 Proposition *Let* C *have pullbacks and regular factorizations. If* F *preserves regular epimorphisms then* $\mathsf{Alg}(F)$ *has regular factorizations created by the forgetful functor.*

Remark We prove a more general statement which concerns *regular cone–factorizations.* Consider a cone, i.e., a (possibly large or empty) collection of morphisms $f_i\colon A \to B_i$ $(i \in I)$. A regular cone–factorization is a regular epimorphism $e\colon A \to A'$ and a collectively monomorphic cone $m_i\colon A' \to B_i$ $(i \in I)$ with $f_i = m_i e$ for all $i \in I$. If a cowellpowered category C has regular factorizations and products, then it has regular cone–factorizations. (In fact, for small cones $f_i\colon A \to B_i$ just form a regular factorization of $\langle f_i\rangle_{i\in I}\colon A \to \prod\limits_{i\in I} B_i$. For large cones use regularly cowellpoweredness on the coimages of $f_i, i \in I$, to reduce to a small cone.) Note that this condition in case of an empty cone simply means: there is a regular epimorphism $A \xrightarrow{e} B$ such that any two maps $r, s\colon B' \to B$ coincide. In Set a regular factorization of a cone $f_i\colon A \to B_i$ can be obtained by factorizing each f_i over A/ρ where ρ is the intersection of the kernels of all f_i.

We prove that whenever C has pullbacks and regular cone-factorizations, then $\mathsf{Alg}(F)$ has regular cone-factorizations created by the forgetful functor.

Proof. Given a cone $f_i\colon (A,\alpha_A) \to (B_i,\alpha_{B_i})$ in $\mathsf{Alg}(F)$ and given a regular cone factorization $f_i = m_i e$ in C with $e\colon A \to C$, there exists

$$\begin{array}{ccccc} FA & \xrightarrow{Fe} & FC & \xrightarrow{Fm} & FB \\ {\scriptstyle \alpha_A}\downarrow & & {\scriptstyle \alpha_C}\vdots & & \downarrow{\scriptstyle \alpha_B} \\ A & \xrightarrow[e]{} & C & \xrightarrow[m_i]{} & B_i \end{array}$$

a unique $\alpha_C\colon FC \to C$ turning e and each m_i into homomorphisms. This follows from the diagonal fill–in property of the regular epimorphism Fe and the (possibly empty) monomorphic cone m_i. Since m_i is monomorphic in C, it follows easily that $m_i\colon (C, \alpha_C) \to (B, \alpha_B)$ is monomorphic in $\mathsf{Alg}(F)$. Let us form a kernel pair $k_1, k_2\colon K \to A$ of e (i.e., a pullback of e and e),

then since e is a regular epimorphism, it is a coequalizer of k_1 and k_2 in C. Now, U creates pullbacks, thus, we have homomorphisms $k_1, k_2\colon (K, \alpha_K) \to (A, \alpha_A)$, and it is easy to see that e is a coequalizer of k_1 and k_2 in $\mathsf{Alg}(F)$. Thus, e is a regular epimorphism in $\mathsf{Alg}(F)$. □

4.10 Corollary *For every endofunctor F of* Set *the category of F–algebras has regular factorizations created by the forgetful functor.*

In fact, regular epimorphisms in Set are just the split epimorphisms which all functors preserve.

4.11 Colimit Theorem *Let* C *have pullbacks and regular cone–factorizations. For every varietor F preserving regular epimorphisms,* $\mathsf{Alg}(F)$ *has colimits of all types existing in* C.

Proof. Let $D\colon \mathsf{D} \to \mathsf{Alg}(F)$ be a diagram with $UDd = A_d$ $(d \in \mathsf{D})$ such that UD has a colimit

$$(A_d \xrightarrow{c_d} C)_{d \in \mathsf{D}}$$

in C. Let $\eta_C\colon C \to UC^\sharp$ be a free F–algebra on C. Form the cone of all homomorphisms $r\colon C^\sharp \to R$ in $\mathsf{Alg}(F)$ with the property that each $Ur \cdot \eta_C \cdot c_d$ is a homomorphism from Dd to R $(d \in \mathsf{D})$. Let

$$C^\sharp \xrightarrow{e} E \xrightarrow{r'} R = C^\sharp \xrightarrow{r} R$$

be a regular factorization of that cone in $\mathsf{Alg}(F)$ (see 4.9). Then each

$$\bar{c}_d = Ue \circ \eta_C \circ c_d\colon Dd \to E \qquad (d \in \mathsf{D})$$

is a homomorphism. In fact, for every r the following square

$$\begin{array}{ccccccccc} FA_d & \xrightarrow{Fc_d} & FC & \xrightarrow{F\eta_C} & FUC^\sharp & \xrightarrow{FUe} & FUE & \xrightarrow{FUr'} & FUR \\ {\scriptstyle \alpha_{A_d}}\downarrow & & & & & & \vdots{\scriptstyle \alpha_E} & & \downarrow{\scriptstyle \alpha_R} \\ A_d & \xrightarrow[c_d]{} & FC & \xrightarrow[\eta_C]{} & UC^\sharp & \xrightarrow[Ue]{} & UE & \xrightarrow[Ur']{} & UR \end{array}$$

commutes because $Ur \cdot \eta_C \cdot c_d$ is a homomorphism. Since the cone of all Ur' is collectively monomorphic, it follows that the left–hand subsquare commutes.

We claim that $(Dd \xrightarrow{\bar{c}_d} (E, \alpha_E))_{d \in \mathsf{D}}$ is a colimit of D in $\mathsf{Alg}(F)$. It is obviously a cocone of D, because (c_d) is a cocone of UD. Let $f_d \colon Dd \to B$ be another cocone of D. Then there exists a unique $f \colon C \to UB$ in C with $Uf_d = f \circ c_d$ $(d \in D)$. The unique homomorphism $r \colon C^\sharp \to B$ with $f = Ur \circ \eta_C$ is a member of the above cone, thus, we have a factorization $r = r' \circ m$. The homomorphism $r' \colon E \to B$ fulfills $f_d = r' \circ \bar{c}_d$ for all d because

$$Uf_d = f \circ c_d = Ur \circ \eta_C \circ c_d = Ur' \circ U\bar{c}_d.$$

And r' is unique: suppose that a homomorphism $h \colon E \to B$ also fulfills $f_d = h \circ \bar{c}_d$ for all d, then we have

$$UhUe\eta_c \circ c_d = Ur' \circ Ue \circ \eta_C \circ c_d \quad (d \in D)$$

which, due to the universal property of colimits, implies

$$Uh \circ Ue \circ \eta_C \;=\; Ur' \circ Ue \circ \eta_C$$

and this implies $h \circ e = r' \circ e$. Since e is an epimorphism, this proves $h = r$. □

4.12 Corollary *Let* C *be a complete, cocomplete, regularly cowellpowered category with regular factorizations. Then for every varietor* F *preserving regular epimorphisms the category of* F*–algebras is cocomplete.*

4.13 Theorem *For every functor* $F \colon \mathsf{Set} \to \mathsf{Set}$ $\mathsf{Alg}(F)$ *is cocomplete, provided that* F *is a varietor.*
The reverse implication holds under the generalized Continuum Hypothesis.

Proof. The implication above is a consequence of 4.12. Set satisfies all the hypotheses, and F preserves regular (= split) epimorphisms automatically.

The reverse implication follows from (Adámek, Koubek and Pohlová 1972, Theorem III.4) which states a necessary and sufficient condition of cocompleteness of categories of bialgebras, $A(F, G)$. The choice of $G = Id_{\mathsf{Set}}$ yields

$$\mathsf{Alg}(F) = A(F, \mathrm{Id}_{\mathsf{Set}})$$

and the theorem states that $\mathsf{Alg}(F)$ has colimits iff F preserves unions or is non–excessive, which means that for arbitrarily large sets X we have $\mathsf{card} FX \leq \mathsf{card} X$. Now every set functor preserving unions is easily seen to be non–excessive. And non–excessive functors are varietors, see (Adámek and Trnková 1990, 4.3). □

4.14 Remark Another important criterion for cocompleteness of categories of algebras follows from the theory of locally presentable categories. Recall that a category C is *locally* λ*–presentable* provided that it is cocomplete and has a set A of λ–presentable objects (i.e., objects A such that the hom–functor of A preserves λ–filtered colimits) such that all objects of C are λ–filtered colimits of objects of A. Examples: $\mathsf{Set}, \mathsf{Vec}_k, \mathsf{Pos}$ are locally $\aleph_0$–presentable, the category of CPO's is locally $\aleph_1$–presentable. But Set^{op} is not locally presentable.

4.15 Theorem *For every accessible endofunctor F of a locally presentable category* C *the category of F–algebras has colimits. In fact,* $\mathsf{Alg}(F)$ *is locally presentable.*

Proof. See (Adámek and Rosický 1994, 2.75). □

Using Lemma 1.4, we obtain the following dualizations of the above results.

4.16 Theorem *The forgetful functor of the category of F–coalgebras creates colimits and all limits preserved by F.*

4.17 Corollary *If* C *is cocomplete then all categories of coalgebras are cocomplete.*

4.18 Corollary *A homomorphism in* $\mathsf{Coalg}(F)$ *is an isomorphism (or an epimorphism) iff its underlying morphism is an isomorphism (an epimorphism, respectively) in* C*. If* C *has cokernel pairs and F perserves equalizers of cokernel pairs, then a homomorphism in* $\mathsf{Coalg}(F)$ *is a regular monomorphisms iff its underlying morphism is a regular monomorphism.*

4.19 Corollary *If* C *is cowellpowered, so are all categories of coalgebras.*

4.20 Proposition (see also (Gumm and Schröder 2001)) *For every covarietor F in* Set *the category of F–coalgebras is complete.*

Proof. If F preserves regular monomorphisms, this follows from the dual of 4.11.

Now recall that regular monomorphisms in Set are just one–to–one functions, and that all except those with empty domains are split (and thus preserved by any functor)! To complete the proof it remains to apply the following result of V. Trnková (see (Adámek and Trnková 1990)).

4.21 Lemma*For every functor $F\colon \mathsf{Set} \to \mathsf{Set}$ which does not preserve monomorphisms there exists a functor $F'\colon \mathsf{Set} \to \mathsf{Set}$ preserving monomorphisms, and such that*

$$\begin{array}{lcll} FX & = & F'X & \text{for all sets } X \neq \emptyset \\ Ff & = & F'f & \text{for all functions } f\colon X \to Y \text{ with } X \neq \emptyset \end{array}$$

and $F\emptyset \neq \emptyset \neq F'\emptyset.$

Proof. See (Adámek and Trnková 1990, III.4.5 – 6). □

Remark Given now F and F' as in this lemma, we have

$$\mathsf{Alg}(F) = \mathsf{Alg}(F')$$

because no F–algebra (and no F'–algebra) has underlying set $\emptyset$. And

$$\mathsf{Coalg}(F) \cong \mathsf{Coalg}(F')$$

because the two categories of coalgebras only differ in the name of the codomain of the unique coalgebra $(\emptyset, \alpha_\emptyset)$.

In particular, if $\mathsf{Coalg}(F')$ is complete, so is $\mathsf{Coalg}(F)$. □

4.22 Remark It is not known whether completeness of $\mathsf{Coalg}(F)$, where F is a Set-functor, implies that F is a covarietor.

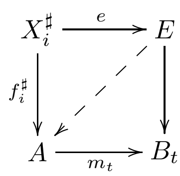
4.23 Remark Dualizing 4.8, we see that for C with pushouts and *coregular factorizations*, i.e., factorizations as an epimorphism followed by a regular monomorphism, the following holds: $\mathsf{Coalg}(F)$ has coregular factorizations created by U whenever F preserves regular monomorphisms. This also generalizes to factorizations of cocones. In Set a coregular factorization of a cocone $f_i\colon B_i \to A$ can be obtained by factorizing each f_i over the subset $X \subseteq A$ which is the union of the images of all f_i. Combined with 4.21 we obtain the following

Corollary *If* $\mathsf{C} = \mathsf{Set}$ *then all categories of coalgebras have coregular factorizations created by the forgetful functor.*

5. Birkhoff Variety (and Covariety) Theorem

5.1 Observation Suppose that F is an endofunctor of a complete category; then the following hold for every equational class V of F–algebras.

1) V is closed under products, i.e., if $A_t \in \mathsf{V}$ for $t \in T$ then $\prod A_t \in \mathsf{V}$.
2) V is closed under subalgebras, i.e., given a monomorphism $m\colon A \to B$ in $\mathsf{Alg}(F)$ with $B \in \mathsf{V}$, then $A \in \mathsf{V}$.
3) Generalizing 1) and 2), V is closed under *monomorphic cones* in $\mathsf{Alg}(F)$. That is, if $m_t\colon A \to B_t$ is a monomorphic family of homomorphisms in $\mathsf{Alg}(F)$, then if $B_t \in \mathsf{V}$ for all t, then $A \in \mathsf{V}$.
   In fact, if $e\colon X_i^\sharp \to E$ is an equation arrow that each B_t satisfies, then A satisfies e, too: for every $f\colon X \to A$ factor $(m_t f)_i^\sharp = m_t f_i^\sharp$ (see 2.12) through e and use the diagonal fill–in



   Note that here the indices might even range over a class T.
4) V is closed under *retract–carried quotient algebras* in $\mathsf{Alg}(F)$, i.e., given a homomorphism $q\colon A \to B$ in $\mathsf{Alg}(F)$ and a morphism $m\colon UB \to UA$ in C with $Uq \circ m = id$, then if $A \in V$, then $B \in V$. In fact, if A satisfies $e\colon X_i^\sharp \to E$ then B also satisfies it: for every $f\colon X \to B$ we have a factorization of $(mf)_i^\sharp\colon X_i^\sharp \to A$ through e, say, $(mf)_i^\sharp = ge$. From 2.12 we get $(qg)e = q(mf)_i^\sharp = (qmf)_i^\sharp = f_i^\sharp$, showing that $f_i^\sharp$ factors through e.

5.2 The Birkhoff Variety Theorem *Let* C *be a complete, regularly cowellpowered category with regular factorizations. For every constructive varietor* F *preserving regular epimorphisms, varieties of* F*–algebras are precisely the full subcategories of* $\mathsf{Alg}(F)$ *closed under*

i. products,
ii. subalgebras, and
iii. retract–carried quotient algebras.

Remark We will also prove that every variety is reflective in $\mathsf{Alg}(F)$ with reflection arrows being regular epimorphisms. Thus, varieties are complete and, in case C is cocomplete, also cocomplete (see 4.11)

Proof. If V is a variety, then it is closed under i. – iii. by 5.1. Conversely, let V be closed under i. – iii.

(a) Observe that $\mathsf{Alg}(F)$ has regular cone–factorizations created by the forgetful functor — see Remark 4.9. Since $\mathsf{Alg}(F)$ is regularly cowellpowered (which is obvious from 4.9), any full subcategory V of $\mathsf{Alg}(F)$ closed under products and subalgebras is closed under monomorphic cones — see (Adámek, Herrlich and Strecker 1990, 16.8). It follows that any such V is reflective in $\mathsf{Alg}(F)$: a reflection of an F–algebra A is obtained from the cone of all morphisms with domain A and codomain in V, $f_t\colon A \to B_t$ $(t \in T)$: let $e\colon A \to A'$ be a regular epimorphism forming a regular factorization of that cone, i.e., we have a monomorphic cone $m_t\colon A' \to B_t$ with $f_t = m_t \circ e$. Since $B_t \in \mathsf{V}$ for each t, we conclude $A' \in \mathsf{V}$. It follows that $e\colon A \to A'$ is a reflection of A.

(b) Let $\mathcal{E}$ be the class of all equation arrows $e\colon X^\sharp \to A'$ where X is an object of C, $A = X^\sharp$ $(= X_i^\sharp$ for some $i)$ is a free F–algebra on X, and $e\colon A \to A'$ is a reflection of A in V. Then we prove

$$\mathsf{V} = \mathsf{Alg}(F, \mathcal{E}).$$

(α) Every algebra $B \in \mathsf{V}$ satisfies each of the equation arrows $e\colon X^\sharp \to A'$ above. In fact, for every morphism $f\colon X \to A$ the unique homomorphism $f^\sharp\colon X^\sharp \to B$ with $f = f^\sharp \circ \eta_X$ factors through e (since $f^\sharp = f_t$ for some t, and $f^\sharp = m_t \circ e$).

(β) Every algebra C satisfying each $e \in \mathcal{E}$ lies in V. Put $X = UC$ and denote by $h\colon X^\sharp \to C$ the unique homomorphism with $h \circ \eta_X = id_X$. For the reflection $e\colon X^\sharp \to A'$ we know that, since C satisfies e, the homomorphism h factors through e, say, $h = ke$. Then $k\colon A' \to C$ is a regular epimorphism carried by a split epimorphism in C (since h is), and thus from $A' \in \mathsf{V}$ we conclude $C \in \mathsf{V}$. □

5.3 *Regular quotients* of A in $\mathsf{Alg}(F)$ are those B in $\mathsf{Alg}(F)$ which admit a regular epimorphism $e\colon A \to B$. For $\mathsf{C} = \mathsf{Set}$ the regular quotients in $\mathsf{Alg}(F)$ are precisely those carried by (split) epimorphisms in Set (i.e., the “quotients” or “homomorphic images” in the usual sense), see 4.3, since coequalizers of kernel pairs in Set are absolute, i.e., preserved by every functor. Thus, the following is a corollary to 5.2 which, for polynomial functors F, specializes to the classical Birkhoff Variety Theorem.

Corollary *For every varietor F over* Set*, varieties of F–algebras are precisely the full subcategories closed in* $\mathsf{Alg}(F)$ *under products, subalgebras and regular quotients.*

5.4 Notation For a class $\mathcal{E}$ of morphisms in a category C we denote by $\mathsf{Inj}\mathcal{E}$ the full subcategory of all objects C injective w.r.t. any member $e\colon A \to A'$ of $\mathcal{E}$ (i.e., for every morphism $f: A \to C$ there exists $f': A' \to C$ with $f = f'e$). We call $\mathsf{Inj}\mathcal{E}$ the *injectivity class* of $\mathcal{E}$.

5.5 Corollary *Under the assumptions of Theorem 5.2, varieties of F–algebras are precisely the injectivity classes $\mathsf{Inj}\mathcal{E}$ in $\mathsf{Alg}(F)$, where $\mathcal{E}$ is a collection of regular quotients of free F–algebras.*

In fact, in the proof of 5.2 we have found, for every variety V, the corresponding collection $\mathcal{E}$, viz, all reflections of free F–algebras in V. Conversely, it is sufficient to verify that $\mathsf{Inj}\mathcal{E}$, for a class $\mathcal{E}$ as above, is closed under

i. products — this is trivial,
ii. subalgebras — this follows from the diagonal fill–in between monomorphisms and regular epimorphisms, and
iii. retract–carried quotients.

To show iii., let $q: R \to Q$ be a homomorphism with $Uq \circ m = id$ for some $m: UQ \to UR$. If R is $\mathcal{E}$–injective, then so is Q: given $e: X^\sharp \to E$ in $\mathcal{E}$ and $h: X^\sharp \to Q$, then for $g = U(mh)\eta_X: X \to UR$ we form the unique homomorphism $g^\sharp: X^\sharp \to R$ with $g = Ug^\sharp \cdot \eta_X$. There exists a homomorphism $k: E \to R$ with $g^\sharp = ke$. We claim that $qk: E \to Q$ is the desired factorization of h through e, i.e., $qke = h: X^\sharp \to Q$. Since both sides are homomorphisms (recall $ke = g^\sharp$), it is sufficient to observe that

$$U(qke) \circ \eta_X = Uq \circ g = Uq \circ Um \circ Uh \circ \eta_X = Uh \circ \eta_X.$$

Using Corollary 3.8 in addition we obtain

5.6 Corollary *Let F be an accessible endofunctor of* Set *and* V *be an equational class of F-algebras. Then there is a single regular quotient e of a free F-algebra $X^\sharp$, such that* $\mathsf{V} = \mathsf{Inj}\{e\}$.

5.7 Remark Recall that an object A is called a *regular projective* provided that it is projective (the dual to the above injective) w.r.t. to all regular epimorphisms.

Example: every free F–algebra $X^\sharp$ for an endofunctor F of Set is a regular projective. In fact, given a regular epimorphism $e: A \to B$ in $\mathsf{Alg}(F)$, then e is a (split) epimorphism in Set, choose $m: UB \to UA$ with $Ue \circ m = id$. For every homomorphism $h: X^\sharp \to B$ form $k = (m \circ Uh \circ \eta_X)^\sharp: X^\sharp \to A$ then by 2.12 we have

$$e \circ k = (Ue \circ m \circ Uh \circ \eta_X)^\sharp = (Uh \circ \eta_X)^\sharp = h.$$

5.8 Corollary *For a varietor F of* Set*, varieties of F–algebras are precisely the injectivity classes of regular quotients of regular projectives in* $\mathsf{Alg}(F)$.

In fact, every variety is such an injectivity class by 5.5 and 5.7. Conversely, let $\mathcal{E}$ be a collection of regular quotients of regular projectives. Then $\mathsf{Inj}\mathcal{E}$ is closed under products and subalgebras (the argument is precisely as in 5.5) and it is closed under regular quotients. The argument for the latter is also analogous to 5.5: let $q: R \to Q$ with $Uq \circ m = id$ and $e: E \to E'$ in $\mathcal{E}$ be given, with E a regular projective. For $h: E \to Q$

we have, since q is a regular epimorphisms (4.10) $k\colon E \to R$ with $h = qk$. From the fact that R is e–injective it now easily follows that so is Q.

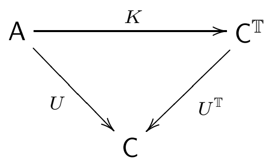
5.9 Remark It might be instructive to show how to obtain, in $\mathsf{Alg}(F)$, from a regular quotient of a regular projective, e, a regular quotient of a free algebra, $\bar{e}$, such that injectivity w.r.t. e and $\bar{e}$ is equivalent. In fact, consider such an epimorphism, $e\colon (D, \alpha_D) \to (E, \alpha_E)$. Since the homomorphism $id^\sharp\colon (D^\sharp, \varphi_D) \to (D, \alpha_D)$ is a regular epimorphism in $\mathsf{Alg}(F)$ and (D, α_D) is regularly projective we have a homomorphism

$$m\colon (D, \alpha_D) \to (D^\sharp, \varphi_D) \quad \text{with} \quad id^\sharp \circ m = id.$$

Choose a pair of homomorphisms u, v with coequalizer e in $\mathsf{Alg}(F)$. Then an algebra (C, α_C) is orthogonal to e iff for every homomorphism $h\colon (D, \alpha_D) \to (C, \alpha_C)$ we have $h \circ u = h \circ v$. This is equivalent to stating that for every homomorphism $k\colon (D^\sharp, \varphi_D) \to (C, \alpha_C)$ we have $k \circ (m \circ u) = k \circ (m \circ v)$: given k, put $h = k \circ m$, and given h, put $k = h \circ id^\sharp$. Thus, if $\bar{e}\colon (D^\sharp, \varphi_D) \to (\bar{E}, \alpha_{\bar{E}})$ denotes a coequalizer of $m \circ u$ and $m \circ v$ in $\mathsf{Alg}(F)$, then injectivity to $\bar{e}$ and e, respectively, is equivalent. (And the former can be substituted by the equations $u_0 = v_0$ obtained by the kernel pair of $\bar{e}$.)

5.10 Remark The above Corollary is much in the spirit of (Herrlich and Ringel 1972) and (Banaschewski and Herrlich 1976).

In the following result we fully characterize varieties as *monadic categories* over C, i.e., such concrete categories $\mathsf{A} \xrightarrow{U} \mathsf{C}$ for which U is a right adjoint and the comparison functor K



of the corresponding monad $\mathbb{T}$ over C is an equivalence functor.

In general, we call concrete categories (A, U) and (A', U') *concretely equivalent* provided that there exists an equivalence functor $K\colon \mathsf{A} \to \mathsf{A}'$ with $U = U'K$.

5.11 Monadicity Theorem *Let C be a cocomplete category. Then monadic categories over C are, up to concrete equivalence, precisely the varieties.*

Proof. I. Let $\mathsf{V} = \mathsf{Alg}(F, \mathcal{E})$ be a variety of F–algebras. By definition, the forgetful functor $U\colon \mathsf{V} \to \mathsf{C}$ is a right adjoint. By Proposition 4.16, U creates all colimits that F preserves, in particular, U creates absolute coequalizers. Consequently, (V, U) is monadic by Beck's Theorem (Mac Lane 1971).

II. For the converse it suffices to show that, for any monad $\mathbb{T} = (T, \eta, \mu)$ on a cocomplete category C, the Eilenberg-Moore category $\mathsf{C}^\mathbb{T}$ of $\mathbb{T}$-algebras coincides with the subcategory $\mathsf{Alg}(T, \mathcal{E})$ of $\mathsf{Alg}(T)$ for a suitable class $\mathcal{E}$ of equation arrows. For doing so consider, for every C-object X, the coproduct

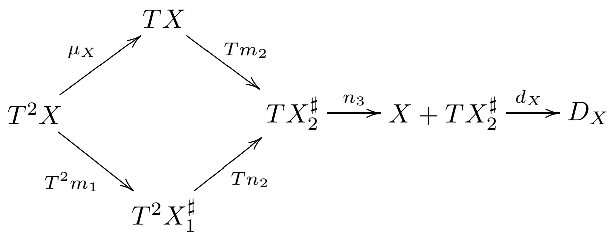
$$X \xrightarrow{m_{i+1}} TX_i^\sharp + X = X_{i+1}^\sharp \xleftarrow{n_{i+1}} TX_i^\sharp.$$

A class $\mathcal{E}_1$ of equation arrows now is defined as follows: for every C-object X let $e_X : X_2^\sharp \to E_X$ be a coequalizer of the pair $m_2, n_2 \circ \eta_{X_1^\sharp} \circ m_1$. A T-algebra (C, α_C) satisfies e_X iff, for every morphism $f : X \to C$, the morphism

$$f_2^\sharp = [\alpha_C \circ T f_1^\sharp, f] : T X_1^\sharp + X \to C$$

satisfies $f_2^\sharp \circ m_2 = f_2^\sharp \circ n_2 \circ \eta_{X_1^\sharp} \circ m_1$ or, equivalently, $f = \alpha_C \circ T f_1^\sharp \circ \eta_{X_1^\sharp} \circ m_1$. Since η is natural, this is equivalent to $f = \alpha_C \circ \eta_C \circ f$ which, for $X = C$ and $f = 1_C$ yields satisfaction of the $\mathbb{T}$-algebra axiom $\alpha_C \circ \eta_C = 1_C$. Conversely, $\alpha_C \circ \eta_C = 1_C$ yields $f = \alpha_C \circ \eta_C \circ f$ by composition with f. Thus, the satisfaction of $\mathcal{E}_1$ is equivalent $\alpha_C \circ \eta_C = 1_C$.

Next we define a class $\mathcal{E}_2$ of equation arrows as follows: for every C-object X let $d_X : X + T X_2^\sharp = X_3^\sharp \to D_X$ be a coequalizer of the pair $n_3 \circ T m_2 \circ \mu_X, n_3 \circ T n_2 \circ T^2 m_1$.



A T-algebra (C, α_C) satisfies d_X iff, for every morphism $f : X \to C$, the morphism

$$f_3^\sharp = [\alpha_C \circ T[\alpha_C \circ T f_1^\sharp, f], f] : T X_2^\sharp + X \to C$$

satisfies $f_3^\sharp \circ n_3 \circ T n_2 \circ T^2 m_1 = f_3^\sharp \circ n_3 \circ T m_2 \circ \mu_X$. This is equivalent to $\alpha_C \circ T\alpha_C \circ T^2 f = \alpha_C \circ T f \circ \mu_X$ or, since μ is natural, to $\alpha_C \circ T\alpha_C \circ T^2 f = \alpha_C \circ \mu_C \circ T^2 f$. Choosing $f = 1_C$ this yields satisfaction of the $\mathbb{T}$-algebra axiom $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$. Conversely, $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$ yields $\alpha_C \circ T\alpha_C \circ T^2 f = \alpha_C \circ \mu_C \circ T^2 f$ by composition with $T^2 f$. Thus, the satisfaction of $\mathcal{E}_2$ is equivalent to $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$.

Chosing $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ one thus gets $\mathsf{C}^{\mathbb{T}} = \mathsf{Alg}(T, \mathcal{E})$. □

5.12 Example (infinitary universal algebra) Generalizing the finitary Ω–algebras of 1.5(2), we introduce here a *large signature* as a collection $\Omega = (\Omega_n)_{n \in \mathsf{card}}$ of sets Ω_n (of n–ary operation symbols) indexed by cardinal numbers n. An Ω*–algebra* is a set A together with an n–ary operation $\omega_A : A^n \to A$ for all $n \in \mathsf{card}$ and $\omega \in \Omega_n$. We denote by Ω–Alg the category of all Ω–algebras and homomorphisms. Well, strictly speaking, this is not a category (having "too many" objects) but a quasicategory: if $\Omega_n \neq \emptyset$ for arbitrarily large cardinal numbers n, then the collection of all Ω–algebras on the set $\{0, 1\}$ is already a proper class. However, we are actually interested in varieties of Ω–algebras, and those are legitimate categories, as we show below.

The concept of Ω–terms over a set X of variables (disjoint from any Ω_n) is introduced as usual: the class $T_\Omega X$ of all terms is the smallest class containing X and containing $\omega(t_i)_{i<n}$ for any $n \in \mathsf{card}, \omega \in \Omega_n$, and $t_i \in T_\Omega X$ for $i < n$. Computation of terms in an Ω–algebra A is also standard: given $f : X \to A$, extend it to $f^\sharp : T_\Omega X \to A$ by $f^\sharp(\omega(t_i)_{i<n}) = \omega_A(f^\sharp(t_i))_{i<n}$. If Ω is small, then $T_\Omega X$ is a free Ω–algebra on X, if Ω is

large, no such free algebra over X exists (even for $X = 1$). By a *variety of* Ω*–algebras* we mean an equationally defined full subcategory V of Ω–Alg having free algebras, i.e., such that the obvious forgetful functor $U\colon \mathsf{V} \to \mathsf{Set}$ has a left adjoint.

Although Ω–Alg is, for Ω large, not expressible by any functor (because $\mathsf{Alg}(F)$ is a category for any F, and Ω–Alg is not), every variety V of Ω–algebras is a monadic category over Set. This follows from Beck's theorem, see the argument in (Mac Lane 1971, VI.8).

By dualizing the above results, using 1.4, we obtain the following result, where *quotients* of a coalgebra A are represented by epimorphisms $e\colon A \to B$ in $\mathsf{Coalg}(F)$, and *coretract-carried subcoalgebras* of A are carried by monomorphisms in $\mathsf{Coalg}(F)$ which split in C. *Regular subcoalgebras* are subcoalgebras represented by regular monomorphisms.

5.13 Theorem (Birkhoff Covariety Theorem.) *Let* C *be a cocomplete, regularly well-powered category with coregular factorizations. For every constructive covarietor* F *preserving regular monomorphisms, covarieties of* F*–coalgebras are precisely the full subcategories of* $\mathsf{Coalg}(F)$ *closed under*

i. *coproducts,*
ii. *quotient coalgebras, and*
iii. *coretract-carried subcoalgebras.*

5.14 *Regular subcoalgebras* of A in $\mathsf{Coalg}(F)$ are those B in $\mathsf{Coalg}(F)$ which admit a regular monomorphism $m\colon B \to A$. For $\mathsf{C} = \mathsf{Set}$ the regular subcoalgebras in $\mathsf{Coalg}(F)$ are precisely those carried by monomorphisms in Set (i.e., the "subcoalgebras" in the usual sense; monomorphisms are more general in Coalg(F)!); this follows from 4.18 if F preserves monomorphisms, since equalizers of cokernel pairs of functions $f\colon B \to A$ with $B \neq \emptyset$ in Set are absolute, i.e., preserved by every functor (and every empty homomorphism $(\emptyset, \alpha_\emptyset) \to (A, \alpha_A)$ is an equalizer of the two coproduct injections of $(A, \alpha_A) + (A, \alpha_A)$). If F does not preserve monomorphisms, apply 4.21.

Thus, since every covariety contains the empty coalgebra, the following corollary is an immediate consequence of 5.13 for functors F which preserve monomorphisms; if F does not preserve monomorphisms, apply again 4.21.

Corollary *For every covarietor* F *over* Set*, covarieties of* F*–coalgebras are precisely the full subcategories closed in* $\mathsf{Coalg}(F)$ *under coproducts, regular subcoalgebras, and quotient coalgebras.*

5.15 Corollary *Under the assumptions of Birkhoff Covariety Theorem, covarieties of* F*-coalgebras are precisely the projectivity classes of regular subcoalgebras of cofree* F*–coalgebras.*

5.16 Corollary *For a covarietor* F *over* Set*, covarieties of* F*–coalgebras are precisely the projectivity classes of regular subobjects of regular injectives in* $\mathsf{Coalg}(F)$*.*

5.17 For the sake of completeness we recall the following result of (Gumm 1999); note that this is *not* the dual of 5.6 (though embeddings of subcoalgebras are precisely the regular monomorphisms)!

Let C *be a covariety of* F*-coalgebras for a bounded* Set*-functor. Then* C *is the projectivity class of a single embedding of a subcoalgebra into a cofree* F*-coalgebra.*

As mentioned in the introduction, this is the type of characterization of covarieties studied by Rutten, Gumm and others.

5.18 Example Given a category Vec_k of vector spaces over a field k denote by F the endofunctor of Vec_k sending a space V to $k \times (\mathsf{V} \otimes \mathsf{V})$. F is a constructive covarietor (see (Porst 2001)). k–coalgebras in the traditional sense (see e.g. (Sweedler 1969)) obviously form a full subcategory of $\mathsf{Coalg}(F)$. By straightforward diagram chasing one can prove that this subcategory satisfies the condition of the Birkhoff Covariety Theorem 5.13 (see (Porst 2001)). Thus *the category of* k*–coalgebras is a covariety of* F*–coalgebras.*

5.19 Comonadicity Theorem *Let* C *be a complete category. Then comonadic categories over* C *are, up to concrete equivalence, precisely the covarieties.*

While the last theorem shows that the dual of a covariety over Set is a variety over Set^{op} it moreover implies the following additional dualization principle:

5.20 Proposition *The dual of a covariety over* Set *is equivalent to a variety over* Set.

Proof. Let C be a covariety over Set with underlying functor U. By means of the contravariant power-set functor $\mathcal{P}'$ the category Set^{op} is monadic over Set. Let $V\colon \mathsf{C} \to \mathsf{Set}$ be the composite of U^{op} and $\mathcal{P}'$. We need to show that V is monadic. Since V has a left adjoint and creates limits it suffices to prove that V creates coequalizers of congruence relations (= kernel pairs). Hence let $r, s\colon (C, \alpha_C) \to (D, \alpha_D)$ be a pair of C-morphisms such that Vr, Vs is a congruence relation and let $q\colon \mathcal{P}'(D) \to X$ be its coequalizer. Since $\mathcal{P}'$ reflects congruence relations and creates their coequalizers there is a unique Set^{op}-morphism $q'\colon D \to X'$ with $\mathcal{P}'(q') = q$ and this is a coequalizer of the congruence relation Ur, Us. If $X' \neq \emptyset$ this will even be a split coequalizer such that U creates from it a coequalizer of r, s. The remaining case $X' = \emptyset$ is trivial: the unique F-coalgebra structure on $\emptyset$ obviously does the job. □

6. Conclusions and Future Research

As mentioned in the introduction, our paper has been inspired by a number of interesting articles devoted to covarieties of coalgebras in the category Set. We have somehow felt that we missed a “beginning of the story”: whereas the concept of a variety has been inspired by examples such as groups and lattices, and developed further through Birkhoff’s classical variety theorem to the more general categorical concepts, in the realm of coalgebras all this has been missing. In the process of asking ourselves what the analogue of equations of universal algebra for coalgebras is we have, to our surprise, observed that the situation is not clear enough for varieties either: the equivalence between (infinitary) varieties and monadicity over Set did not seem to have an adequate formalization over other categories. This has led us to the concept of equation based on the free–algebra construction for a functor F that, in fact, does not have free algebras. The fact that, then, the slogan

$$\text{varietal} = \text{monadic}$$

holds for all cocomplete categories seems to support our approach.

To dualize the story, and to get the concepts of coequation and covariety so that the slogan

$$\text{covarietal} = \text{comonadic}$$

holds in complete categories has been an easy way to the original goal of our paper. But then we were surprised of how scarce the concrete examples of covarieties in literature were. In the present paper we have shown a number of natural examples of covarieties. Some of them, e.g., reflexivity and symmetry for transition systems, are somehow reminiscent of modal logic, but we have not pursued this in any way.

In order to prove an analogue of Birkhoff's (Co) Variety Theorem much more than just limits and colimits needs to be assumed. But we hope that the side conditions presented in our paper are close to the minimum of what one should request. It would be worthwhile to supplement the results above with counter–examples demonstrating that this or that of the side conditions is indispensible.

Our paper is an extension and refinement of the extended abstract (Adámek and Porst 2001) except that the last part of that abstract, in which

$$\text{accessible} = \text{bounded}$$

is proved for all set functions has been omitted. The reason is that we are preparing a paper (Adámek and Porst) devoted to the properties of the categories of coalgebras on accessible functors, where this result fits much better. (The main property is that if F is λ–accessible then F–algebras form a λ–accessible category; this holds "almost" but not quite).

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